Stat 260/CS 294 Homework Assignment 2 (due March 1)

1. Find a transform of θ, $\eta = g(\theta)$, such that the Fisher information $I(\eta)$ is constant (and therefore the Jeffreys prior is constant) for:
   (a) the Poisson distribution, $\mathrm{Poi}(\theta)$;
   (b) the gamma distribution, $\mathrm{Ga}(\alpha, \theta)$, with $\alpha = 1, 2, 3$;
   (c) the binomial distribution, $\mathrm{Bin}(n, \theta)$; and
   (d) the Maxwell distribution, $\mathrm{Max}(\theta)$: $p(x|\theta) \propto \theta^{3/2} x^2 e^{-\frac{\theta x}{2}}, \quad x \geq 0, \theta > 0$.

2. Consider n random variables $x_1, \ldots, x_n$, such that the first ξ of these variables has an $\mathrm{Exp}(\eta)$ distribution and the $n - \xi$ other have an $\mathrm{Exp}(c\eta)$ distribution, where c is known and ξ takes its values in $\{1, 2, ..., n-1\}$.
   (a) Let $z = (z_2, \ldots, z_n)$ where $z_i = x_i/x_1$. Using Bayes' theorem, give the shape of the posterior distribution $p(\xi|x)$ when we use the prior $p(\xi, \eta) = p(\xi)$ and show that it only depends on z.
   (b) Directly (without using Bayes' theorem) show that the distribution of z, $p(z|\xi, \eta)$, only depends on ξ; i.e., $p(z|\xi, \eta) = p(z|\xi)$.
   (c) Show that the posterior distribution $p(\xi|x)$ computed in (a) cannot be written as a product of the posterior distribution for $p(z|\xi)$ as computed in (b) and $p(\xi)$, whatever $p(\xi)$ is, even though we showed in part (a) that $p(\xi|x)$ only depends on z. How do you explain this?
   (d) Show that the paradox does not occur when we use the scale invariant prior $p(\xi, \eta) = p(\xi)\eta^{-1}$.

3. Consider n observations from a multinomial distribution where each observation belongs to one of 3 categories, with probabilities θ_1, θ_2, and $1 - \theta_1 - \theta_2$. The probability of observing r_1, r_2 and $n - r_1 - r_2$ observations from the three categories is given by: $p(r_1, r_2|n, \theta_1, \theta_2) = \frac{n!}{r_1! r_2! (n - r_1 - r_2)!} \theta_1^{r_1} \theta_2^{r_2} (1 - \theta_1 - \theta_2)^{n - r_1 - r_2}$ for $0 \leq r_1 + r_2 \leq n$.

   We are interested in the ratio $\phi = \frac{\theta_1}{\theta_2}$ while we treat $\lambda = \theta_2$ as a nuisance parameter. Assume that the joint distribution of (λ, ϕ) is asymptotically normal.
   (a) Derive the covariance matrix of the asymptotic posterior distribution of (λ, ϕ). Also derive the asymptotic marginal standard deviation of ϕ and the asymptotic conditional standard deviation of λ given ϕ.
   (b) Show that the conditional reference prior is
   $$\pi(\lambda|\phi) \propto (\lambda(1+\phi))^{-\frac{1}{2}} (1 - \lambda(1+\phi))^{-\frac{1}{2}}, \quad 0 < \lambda < \frac{1}{1+\phi}$$
   (c) Show that the marginal reference prior is
   $$\pi(\phi) \propto \phi^{-\frac{1}{2}} (1+\phi)^{-1}$$
   (d) Derive the marginal posterior
   $$\pi(\phi|r_1, r_2, n)$$
   What does the form of the posterior say about the dependence of ϕ on the observations?