1. Show that for all $1 \leq p < \infty$, we have:

$$H_p(\epsilon, Q, F) \leq H_{p,B}(\epsilon, Q, F)$$

for all $\epsilon$. Show that if $Q$ is a probability measure, we have:

$$H_{p,B}(\epsilon, Q, F) \leq H_\infty(\epsilon/2, F).$$

2. Show that the Cramér-von Mises statistic, $n \int (F_n - F)^2 dF$, is a continuous function of the empirical process (with respect to the supremum norm). Show that the distribution of this statistic is independent of $F$, for all continuous $F$.

3. Let $H \subset L^2(P)$ be a function class, where $P$ is some measure defined on the real line. Now consider the following class of additive models:

$$\mathcal{F}_{d,H} = \{ f : \mathbb{R}^d \to \mathbb{R} | f(X_1, X_2, ..., X_d) = \sum_{j=1}^{d} h_j(X_j) \text{ where } h_j \in H \text{ for each } j \},$$

where $X_j, j = 1, 2, ..., d$ are i.i.d and distributed according to the measure $P (X = (X_1, X_2, ..., X_d) \sim P^d)$. Prove the following upper bound for the covering entropy on $\mathcal{F}_{d,H}$ with respect to the norm induced by $L^2(P^d)$:

$$\log N_2(\epsilon; \mathbb{P}; \mathcal{F}_{d,H}) \leq d \log N_2(\frac{\epsilon}{\sqrt{d}}; \mathbb{P}; H).$$

Would the bound hold if the $X_j$’s were identically distributed but not necessarily independent?

4. Derive the VC dimension for the following class of sets in $\mathbb{R}^n$:

$$\{ x \in S_{a_1,a_2,...,a_n,b} \mid \sum_{i=1}^{n} a_i x_i \leq b; \text{ where } a_1, a_2, ..., a_n, b \in \mathbb{R} \}.$$

5. Consider the function class $\mathcal{F} = \{ f_\alpha : \mathbb{R} \to \mathbb{R} | f(x) = \alpha x \text{ where } |\alpha| \leq R \}$. Let $\mathbb{P}$ denote the standard Gaussian measure (i.e. $X \sim \mathbb{P}$ means that $X \sim \mathcal{N}(0,1)$).

(a) Determine $\log N_1(\epsilon; \mathbb{P}; \mathcal{F})$ exactly.

(b) Prove that

$$\mathbb{P}\left( \log N_1(\epsilon; \mathbb{P}; \mathcal{F}) \geq \log N_1(\frac{\epsilon}{2}; \mathbb{P}; \mathcal{F}) \right) \to 0.$$

Hint: You may find the following bound useful:

For a standard $\chi^2$ random variable with $d$ degrees of freedom $Z_d$, we have the following tail bound:

$$\mathbb{Q}[Z_d \geq d(1 + t)] \leq \exp\left( -\frac{3}{16} dt^2 \right).$$

(c) Now use Theorem 24 from Pollard (page 25) to prove a uniform law of large numbers for $\mathcal{F}$. 