**Stat210B: Theoretical Statistics** **Lecture Date: March 13, 2007**

# Quadratic Mean Differentiability and Contiguity

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**Lemma 1 (Sufficient conditions for QMD).** *Fix $\theta_0 \in \Theta \backslash \partial\Theta$.*

- *assume $p_\theta^{1/2}$ is an absolutely continuous function of $\theta$ in some neighbourhood of $\theta_0$ for $\mu$-almost all $x$.*
- *assume the derivative $p'_\theta$ at $\theta_0$ exists for $\mu$-almost all $x$.*
- *asume that the Fisher information exists and is continuous at $\theta_0$.*

*Then $p_\theta$ is QMD.*

**Example 2.**
- exponential families
- location families $p_\theta(x) = f(x-\theta)$ where $f^{1/2}$ is absolutely continuous and $f'$ exists almost everywhere. Note: $I = -\mathbb{E}\left(\left(\frac{f'(x-\theta)}{f(x-\theta}\right)^2\right)$ exists!
- e.g. the Cauchy location model $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$
- e.g. the Laplace location model $f(x) = \frac{1}{2}e^{-|x|}$

**Example 3.** A family that is not QMD: The $Uniform[0,\theta]$ distributions. Proof:

$$\int \left(\sqrt{n}\left(p^{1/2}_{\theta+h/\sqrt{n}}(x) - p_\theta^{1/2}(x)\right)\right)^2 \geq n\left(\int_\theta^{\theta+h/\sqrt{n}} \frac{1}{\theta+h/\sqrt{n}}dx\right) \to \infty \; as \; n \to \infty$$

Reminder of the definition of absolute continuity: $Q << P$ iff $P(A) = 0 \Rightarrow Q(A) = 0, \quad \forall A$

**Theorem 4 (Radon Nikodym).**

$$Q << P \Rightarrow \exists g : Q(A) = \int_A g \, dP$$

*Write:* $g =: \frac{dQ}{dP}$
*Likelihood ratio=Radon Nikodym derivative=g*

**Lemma 5 (Lemma 6.2, van der Vaart (1998) p86).** *Let $P$ and $Q$ have densities $p$ and $q$ w.r.t. $\mu$. Then we can write:*

$$Q = Q^a + Q^\perp, \quad where \tag{1}$$
$$Q^a(A) = Q(A \cap p > 0) \tag{2}$$
$$Q^\perp(A) = Q(A \cap p = 0) \tag{3}$$

*With this notation we have:*

1. *$Q^a << P, Q^{\perp} \perp P$*
2. *$Q^a(A) = \int_A (q/p) dP$ for all measurable sets $A$*
3. *$Q << P$ iff $Q(p=0) = 0$ iff $\int (q/p) dP = 1$*

**Corollary 6 (change of measure).** *Let $Q << P$, let $P^{X,V}$ denote the law of the pair $(X, V) = (X, \frac{dQ}{dP})$ under $P$. Then*

$$Q(X \in B) = \mathbb{E}_P \left[ 1_B(X) \frac{dQ}{dP} \right] = \int_{B \times \mathbb{R}} v dP^{X,V}(x, v)$$

**Definition 7 (Contiguity).** $Q_n$ is contiguous with respect to $P_n$, in symbols $Q_n \triangleleft P_n$, if $\forall \{A_n\} : P_n(A_n) \to 0$ implies $Q_n(A_n) \to 0$

**Example 8 (Absolute continuity does not imply contiguity).** Let $P_n = N(0,1)$, $Q_n = N(\xi_n, 1)$, $\xi_n \to \infty$, $A_n = \{x : |x - \xi_n| < 1\}$. Then $Q_n << P_n$ but not $Q_n \triangleleft P_n$.

**Example 9 (Contiguity does not imply absolute continuity).** Let $P_n = Uni[0,1]$, $Q_n = Uni[0, \theta_n]$, $\theta_n \to 1$, $\theta_n > 1$. Then $Q_n \triangleleft P_n$ but not $Q_n << P_n$.

$$\mathbb{E}_{P_n} \left[ \frac{dQ_n}{dP_n} \right] = \int \frac{q_n}{p_n} p_n d\mu = \int_{p_n > 0} q_n d\mu = Q_n\{x : p_n > 0\} \leq 1$$

Hence $\frac{dQ_n}{dP_n}$ is uniformly tight. Prohorov's theorem then implies that for all subsequences of $\frac{dQ_n}{dP_n}$ there exists a further weakly converging subsequence. As the following lemma shows, the limit points determine contiguity.

**Lemma 10 (Le Cam's first lemma, Lemma 6.4, van der Vaart (1998) p88).** *The following statements are equivalent:*

- *$Q_n \triangleleft P_n$*
- *If $\frac{dQ_n}{dP_n} \overset{P_n}{\rightsquigarrow} V$ along a subsequence, then $\mathbb{E}V = 1$*
- *$T_n \overset{P_n}{\to} 0$ implies $T_n \overset{Q_n}{\to} 0$*

**Corollary 11 (Asymptotic log normality, van der Vaart (1998) p89).** *Suppose $\frac{dQ_n}{dP_n} \overset{P_n}{\rightsquigarrow} \exp\left[N(\mu, \sigma^2)\right]$. Then $Q_n \triangleleft \triangleright P_n$ iff $\mu = -\frac{1}{2}\sigma^2$.*

*Proof.* Idea of proof: Let $Z \sim N(\mu, \sigma^2)$. By Le Cam's first lemma, we need $\mathbb{E}e^Z = 1$ for $Q_n \triangleleft P_n$. But $\mathbb{E}e^Z = \exp(\mu + \frac{1}{2}\sigma^2)$. (Characteristic functions!) □

**Example 12.** Let $P_n = N(0,1)$, $Q_n = N(\xi_n, 1)$. Then $\frac{dQ_n}{dP_n} = \exp[\xi_n x - \frac{1}{2}\xi_n^2]$. This converges if $\xi_n \to \xi$ with $|\xi| < \infty$ which yields $\exp[\xi x - \frac{1}{2}\xi^2]$ in the limit, hence we get contiguity for $|\xi| < \infty$

**Example 13.** Let $X_i \overset{i.i.d.}{\sim} N(\xi, 1)$, let $P_n$ be the joint distribution for $\xi = 0, i = 1 \ldots n$ and $Q_n$ for $\xi = \xi_n, i = 1 \ldots n$. Then $\log \frac{dQ_n}{dP_n} = \xi_n \sum_i X_i - \frac{n\xi_n^2}{2}$, hence $\frac{dQ_n}{dP_n} \sim \exp\{N(-\frac{n\xi_n^2}{2}, n\xi_n^2)\}$. Therefore we need $\xi_n = O(n^{-1/2})$.

**Example 14 (QMD families, Theorem 7.2, van der Vaart (1998) p94).**

$$\log \frac{dP_{\theta_0 + h/\sqrt{n}}}{dP_{\theta_0}} = \frac{1}{\sqrt{n}} \sum_i h^t \dot{l}_{\theta_0}(X_i) - \frac{1}{2} h^t I_{\theta_0} h + o_{P_{\theta_0}}(1)$$

We get mean $= -\frac{1}{2}$ variance in the limit, i.e. for qmd families $P_{\theta_0 + h/\sqrt{n}} \triangleleft P_{\theta_0}$.

## References

van der Vaart, A. W. (1998). *Asymptotic Statistics*. Cambridge University Press, Cambridge.