Problem Set 3 for CS 170

Problem 1 [Transformed Costs]

Suppose we are given an instance of the Shortest Path problem with source vertex s on a directed graph G. Assume that all edges costs are positive and distinct. Let P be a minimum cost path from s to t. Now suppose that we replace each edge cost c_e by its square, c_e^2, thereby creating a new instance of the problem with the same graph but different costs. True or false: Is P still a minimum-cost $s-t$ path for this new instance? If true give a proof; if false give a counterexample.

Problem 2 [BFS and DFS]

We have a undirected, connected graph $G = (V, E)$ and a specific vertex $u \in V$. Suppose that we compute a depth-first search tree rooted at u and obtain a tree T. Suppose we then compute a breadth-first search tree rooted at u, and obtain the same tree T. Prove that $G = T$. (In other words, if T is both a depth-first search tree and a breadth-first search tree rooted at u, then G cannot contain any edges that do not belong to T).

Problem 3 [Separators]

In an undirected graph, a set of nodes S is called a *minimal u–v separator* if the removal of S cuts all paths between u and v and the removal of any proper subset of S does not. Given an efficient algorithm to find a minimal u–v separator.

Problem 4 [Running]

To get in shape, you have decided to start running to the university. You want a route that goes entirely uphill and then downhill so that you can work up a sweat going uphill and then get a nice breeze at the end of your run as you run faster downhill. Your run will start at home and end at the university. You have a map detailing the roads; there are m road segments (i.e., a segment between two intersections) and n intersections. Each road segment has a positive length, and each intersection has a distinct elevation.

1. Assuming that every road segment is either uphill or downhill, give an efficient algorithm to find the shortest route that meets your specifications and analyze its running time.

2. Now assume that some roads may be level (i.e., both intersections at the end of the road segments are at the same elevation). Level segments can be taken at any point during the run. Give an effient algorithm to solve the problem.