Problem Set 2 for CS 170

Problem 1 [Adjacency Matrices]

Let $A$ denote the adjacency matrix of a directed graph $G = (V, E)$ (a binary matrix such that $a_{ij} = 1$ iff $(i, j) \in E$).

a Prove that the $(i, j)$th entry of the matrix $A^k$ is the count of the number of paths from node $i$ to node $j$ that have exactly $k$ edges.

b Under the assumption that $G$ is acyclic, give an efficient algorithm for obtaining the ancestor matrix (a binary matrix $B$ such that $b_{ij} = 1$ iff $i$ is an ancestor of $j$) from the adjacency matrix $A$.

Problem 2 [Cliques]

A clique is a set of vertices in an undirected graph $G = (V, E)$ such that every pair of vertices in the set is connected by an edge. Given a clique $C \subseteq V$, show that all the vertices in $C$ lie in a single root-to-leaf path in a depth-first search tree. Must the clique vertices be adjacent to one another in the path?

Problem 3 [Bipartite Graph Test]

An undirected graph $G = (V, E)$ is bipartite if and only if its vertices can be partitioned into sets $V_1$ and $V_2$ such that every edge has one vertex in $V_1$ and the other in $V_2$. Give a linear time algorithm based on depth-first search for determining if a given graph is bipartite.

Problem 4 [Semiconnectedness]

A directed graph $G = (V, E)$ is said to be semiconnected if for any pair $(u, v)$ of vertices there is a path either from $u$ to $v$, or from $v$ to $u$ (or both).

a Give an efficient algorithm to determine whether a given graph is semiconnected.

b Show that $G$ is semiconnected if and only if the DAG of its strongly connected components have a unique topologically sorted order.