Problem Set 12 for CS 170

Problem 1 [In NP?]

Give verification algorithms to show that the following problems are in **NP**. Formulate each problem as a decision problem first, if necessary.

(a) Longest Path: Given an undirected graph $G = (V, E)$ and nodes $u, v \in V$, what is the longest simple path between u and v?

(b) Graph Coloring: Given an undirected graph $G = (V, E)$, what is the minimum number of colors for which one can assign a color to each vertex so that no two adjacent vertices have the same color.

Problem 2 [Dominating Sets]

You're designing a networked database system in which each workstation in your cluster needs access to a core database. You want to ensure that no workstation in the cluster is more than one hop away from a machine that has a copy of the database.

Based on this setup, we define the *Dominating Set Problem*: given a graph G, and a number k, is there a way to place k copies of the core database at k different nodes so that every node either has a copy of the database or is connected by a direct link to a node that has a copy of the database?

Prove that the Dominating Set Problem is NP complete.

Problem 3 [Maximum Cliques]

Consider the Maximum Clique problem restricted to graphs where every vertex has degree at most 3, call it MC3.

(a) Prove that MC3 is in NP.

(b) What is wrong with the following proof of NP-completeness for MC3?

> We know that the Maximum Clique problem in general graphs (abbreviated MC) is NP-complete, so it is enough to present a reduction from MC3 to MC. Given a graph of degree at most 3 G and a parameter k, the reduction leaves the graph and the parameter unchanged: clearly the output of the reduction is a possible input for the MC problem. Furthermore, the answer is YES for the MC problem if and only if it was YES for the MC3 problem. This proves the correctness of the reduction and the NP-completeness of MC3.

(c) It is true that that the Vertex Cover problem remains NP-complete even when restricted to graphs where every vertex has degree at most 3 (call this problem VC3). What is wrong with the following proof of NP-completeness for MC3?

We present a reduction from VC3 to MC3. Given a graph $G = (V, E)$ of maximum degree 3 and a parameter k, we give as output the same graph G and the parameter $|V| - k$. The reduction creates a possible input for MC3. Furthermore, a subset $C \subseteq V$ is a vertex cover in G if and only if the complementary set $V - C$ is a clique in G. Therefore G has a vertex cover of size $\leq k$ if and only if it has a clique of size $\geq |V| - k$. This proves the correctness of the reduction and the NP-completeness of MC3.

(d) Describe and analyze a $O(|V|^4)$ algorithm for MC3.

Problem 4 [Approximate Subset Sums]

Suppose you are given a set of positive integers $A = \{a_1, a_2, \ldots, a_n\}$ and a positive integer B. A subset $S \subseteq A$ is called *feasible* if the sum of the numbers in S does not exceed B. The sum of the numbers in S will be called the *total sum* of S. You would like to select a feasible subset S whose total sum is as large as possible.

(a) Consider a greedy algorithm which goes through the set A in order and adds $\{a_i\}$ to the subset that it is building if the total of the elements added thus far (including a_i) is not greater than B. Give an instance in which the total sum of the set S returned by this algorithm is less than half the total sum of some other feasible subset of A.

(b) Give a polynomial-time approximation algorithm for this problem with the following guarantee: It returns a feasible set $S \subseteq A$ whose total sum is at least half as large as the maximum total sum of any feasible set $S' \subseteq A$. Your algorithm should run in time at most $O(n \log n)$.