Problem Set 10 for CS 170

Problem 1 [Changing a Capacity]

Suppose you are given a directed graph $G = (V, E)$, with a positive integer capacity c_e on edge e, a designated source $s \in V$ and a designated sink $t \in V$. You are also given an integer maximum $s - t$ flow in G, defined by a flow value f_e on each edge e.

Now suppose we pick a specific edge e and increase its capacity by one unit. Show how to find a maximum flow in the resulting capacitated graph in time $O(m + n)$, where m is the number of edges in G and n is the number of nodes.

Problem 2 [Mobile Connectivity]

Consider a set of mobile computing clients in a certain town who each need to be connected to one of several possible *base stations*. We'll suppose there are n clients, with the position of each client specified by its (x, y) coordinates in the plane. There are also k base stations; the position of each of these is specified by (x, y) coordinates as well.

For each client, we wish to connect it to exactly one of the base stations. Our choice of connections is constrained in the following ways. There is a *range parameter* r—a client can only be connected to a base station that is within distance r. There is also a *load parameter* L—no more than L clients can be connected to any single base station.

Your goal is to design a polynomial-time algorithm for the following problem. Given the positions of a set of clients and a set of base stations, as well as the range and load parameters, decide whether every client can be connected simultaneously to a base station, subject to the range and load conditions in the previous paragraph.

Problem 3 [Flood Response]

Consider the following scenario. Due to large-scale flooding in a region, paramedics have identified a set of n injured people distributed across the region who need to be rushed to hospitals. There are k hospitals in the region, and each of the n people needs to be brought to a hospital that is within a half-hour's driving time of their current location.

At the same time, one doesn't want to overload any one of the hospitals by sending too many patients its way. The paramedics are in the touch by cell phone, and they want to collectively work out whether they can choose a hospital for each of the injured people in such a way that the load on the hospitals is *balanced*: Each hospital receives at most $\lceil n/k \rceil$ people.

Give a polynomial-time algorithm that takes the given information about the people's locations and determines whether this is possible.

Problem 4 [Generalizations of Max-Flow]

The max-flow problem can be generalized in a number of directions:

(a) There are many sources and many sinks, and we wish to maximize the total flow from all sources to all sinks.

(b) Each edge has not only a capacity, but also a *lower bound* on the flow it must carry.

(c) The outgoing flow from each node v is not the same as the incoming flow, but is smaller by a factor of $(1 - \epsilon_v)$, where ϵ_v is a loss coefficient associated with the node v.

(d) Each edge has a cost per unit flow associated with it, and we must find among all flows of maximum value the one that minimizes the total cost.

In each case, show how to solve the more general problem by (1) reducing it to the original max-flow problem whenever possible, or (2) by reducing it to linear programming in the remaining cases.