CS-184: Computer Graphics

Lecture #12: Curves and Surfaces

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V2014-F-12-1.0

Today

- General curve and surface representations
- Splines and other polynomial bases

Geometry Representations

- Constructive Solid Geometry (CSG)
- Parametric
  - Polygons
  - Subdivision surfaces
- Implicit Surfaces
- Point-based Surface

- Not always clear distinctions
  - *i.e.* CSG done with implicits

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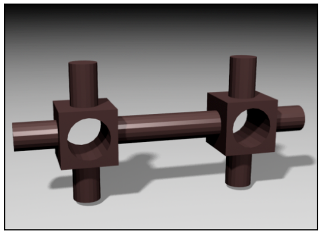
Geometry Representations



Object made by CSG
Converted to polygons

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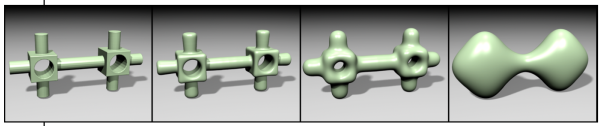
Geometry Representations

Object made by CSG
Converted to polygons
Converted to implicit surface



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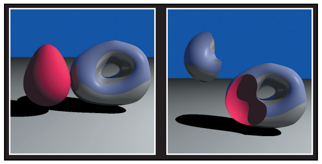
Geometry Representations



CSG on implicit surfaces

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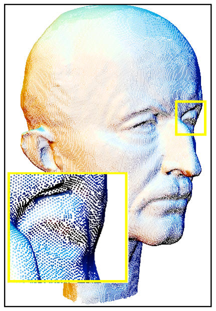
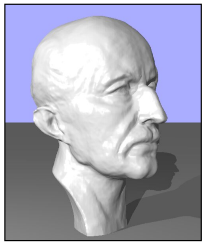
Geometry Representations



Point-based surface descriptions



Ohtake, ***et al.***, SIGGRAPH 2003

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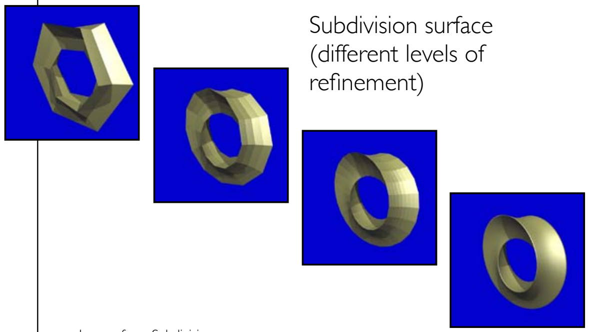
Geometry Representations

Subdivision surface (different levels of refinement)



Images from Subdivision.org

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Geometry Representations

- Various strengths and weaknesses
  - Ease of use for design
  - Ease/speed for rendering
  - Simplicity
  - Smoothness
  - Collision detection
  - Flexibility (in more than one sense)
  - Suitability for simulation
  - *many others...*

Parametric Representations

Curves: $\boldsymbol{x} = \boldsymbol{x}(u)$ $\quad \boldsymbol{x} \in \Re^n$ $\quad u \in \Re$

Surfaces: $\boldsymbol{x} = \boldsymbol{x}(u, v)$ $\quad \boldsymbol{x} \in \Re^n$ $\quad u, v \in \Re$
$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{u})$ $\quad \boldsymbol{u} \in \Re^2$

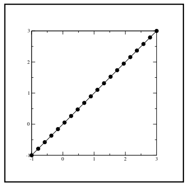
Volumes: $\boldsymbol{x} = \boldsymbol{x}(u, v, w)$ $\quad \boldsymbol{x} \in \Re^n$ $\quad u, v, w \in \Re$
$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{u})$ $\quad \boldsymbol{u} \in \Re^3$

and so on...

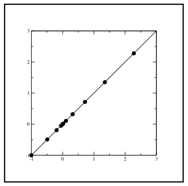
Note: a vector function is really $\boldsymbol{n}$ scalar functions

Parametric Rep. Non-unique

- Same curve/surface may have multiple formulae



$\boldsymbol{x}(u) = [u, u]$



$\boldsymbol{x}(u) = [u^3, u^3]$

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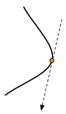
Simple Differential Geometry

- Tangent to curve

$$\boldsymbol{t}(u) = \left.\frac{\partial \boldsymbol{x}}{\partial u}\right|_u$$

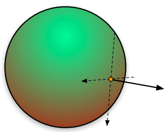


- Tangents to surface

$$\boldsymbol{t}_u(u,v) = \left.\frac{\partial \boldsymbol{x}}{\partial u}\right|_{u,v} \qquad \boldsymbol{t}_v(u,v) = \left.\frac{\partial \boldsymbol{x}}{\partial v}\right|_{u,v}$$

- Normal of surface

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{t}_u \times \boldsymbol{t}_v}{||\boldsymbol{t}_u \times \boldsymbol{t}_v||}$$



- Also: curvature, curve normals, curve bi-normal, ***others...***
- Degeneracies: $\partial \boldsymbol{x}/\partial u = 0$ or $\boldsymbol{t}_u \times \boldsymbol{t}_v = 0$

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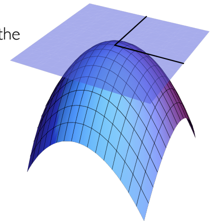
Tangent Space

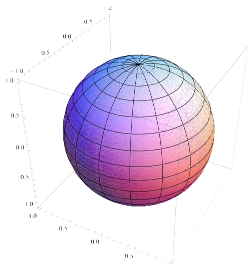
- The *tangent space* at a point on a surface is the vector space spanned by

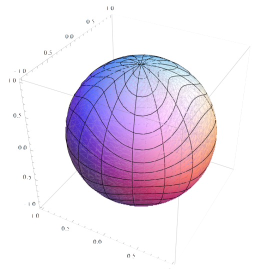
$$\frac{\partial \mathbf{x}(\mathbf{u})}{\partial u} \qquad \frac{\partial \mathbf{x}(\mathbf{u})}{\partial v}$$

  - Definition assumes that these directional derivatives are linearly independent.
  - Tangent space of surface may exist even if the parameterization is bad
- For surface the space is a plane
  - Generalized to higher dimension manifolds



Non Orthogonal Tangents



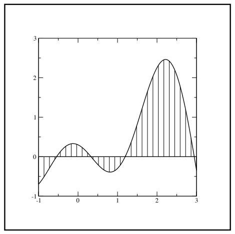


$$\begin{bmatrix} \cos(\theta 2\pi)\cos(\phi pi/2) \\ \sin(\theta 2\pi)\cos(\phi\pi/2) \\ \sin(\phi\pi/2) \end{bmatrix} \qquad \begin{bmatrix} \cos(2\pi\theta)\cos\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right) \\ \cos\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right)\sin(2\pi\theta) \\ \sin\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right) \end{bmatrix}$$

$\theta \in [0..1] \quad \phi \in [-1..1]$

Discretization

- Arbitrary curves have an uncountable number of parameters



i.e. specify function value at all points on real number line

Discretization

- Arbitrary curves have an uncountable number of parameters
- Pick ***complete*** set of basis functions $x(u) = \sum_{i=0}^{\infty} c_i \phi_i(u)$
  - Polynomials, Fourier series, ***etc.***
- Truncate set at some reasonable point

$$x(u) = \sum_{i=0}^{3} c_i \phi_i(u) = \sum_{i=0}^{3} c_i\, u^i$$

- Function represented by the vector (list) of c_i
- The c_i may themselves be vectors $\boldsymbol{x}(u) = \sum_{i=0}^{3} \boldsymbol{c}_i \phi_i(u)$

Polynomial Basis

- Power Basis

$$x(u) = \sum_{i=0}^{d} c_i \, u^i$$

$$x(u) = \boldsymbol{C} \cdot \boldsymbol{\mathcal{P}}^d \qquad \boldsymbol{C} = [c_0, c_1, c_2, \ldots, c_d] \quad \boldsymbol{\mathcal{P}}^d = [1, u, u^2, \ldots, u^d]$$

The elements of $\boldsymbol{\mathcal{P}}^d$ are ***linearly independant***

i.e. no good approximation

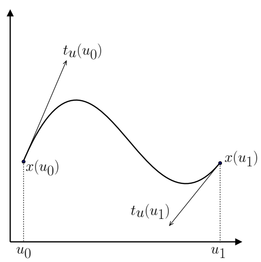
$$u^k \not\approx \sum_{i \neq k} c_i \, u^i$$

Skipping something would lead to bad results... odd stiffness

Specifying a Curve

Given desired values (constraints) how do we determine
the coefficients for cubic power basis?

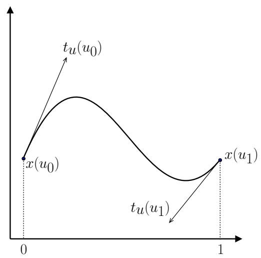
For now, assume
$u_0 = 0 \quad u_1 = 1$



Specifying a Curve

Given desired values (constraints) how do we determine
the coefficients for cubic power basis?

$$x(0) = c_0 \quad = x_0$$
$$x(1) = \Sigma c_i \quad = x_1$$
$$x'(0) = c_1 \quad = x'_0$$
$$x'(1) = \Sigma\, i\, c_i = x'_1$$

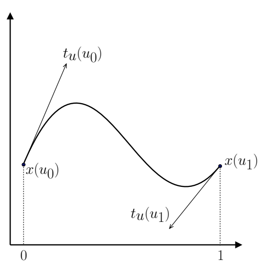


Specifying a Curve

Given desired values (constraints) how do we determine
the coefficients for cubic power basis?

$$\begin{bmatrix} x_0 \\ x_1 \\ x'_0 \\ x'_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\mathbf{p} = \mathbf{B} \cdot \mathbf{c}$$

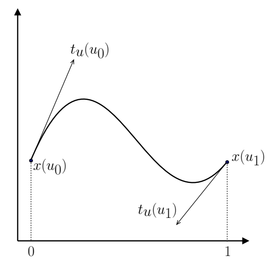


Specifying a Curve

Given desired values (constraints) how do we determine
the coefficients for cubic power basis?

$$\mathbf{c} = \beta_{\text{H}} \cdot \mathbf{p}$$

$$\boldsymbol{\beta}_{\text{H}} = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$



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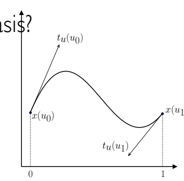
Specifying a Curve

Given desired values (constraints) how do we determine
the coefficients for cubic power basis?

$$\mathbf{c} = \beta_{\text{H}} \cdot \mathbf{p}$$

$$x(u) = \mathcal{P}^3 \cdot \mathbf{c} = \boxed{\mathcal{P}^3 \boldsymbol{\beta}_{\text{H}}}\mathbf{p}$$

$$= \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix} \mathbf{p}$$



$$\boldsymbol{\beta}_{\text{H}} = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

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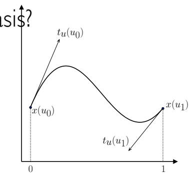
Specifying a Curve

Given desired values (constraints) how do we
determine
the coefficients for cubic power basis?

$$\mathbf{c} = \beta_{\scriptscriptstyle H} \cdot \mathbf{p}$$



$$x(u) = \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix} \mathbf{p}$$

$$x(u) = \sum_{i=0}^{3} p_i b_i(u)$$

Hermite basis functions

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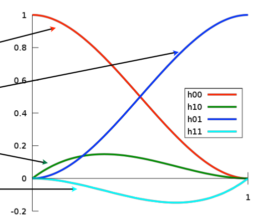
Specifying a Curve

Given desired values (constraints) how do we determine
the coefficients for cubic power basis?

$$x(u) = \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix} \mathbf{p}$$



$$x(u) = \sum_{i=0}^{3} p_i b_i(u)$$

Hermite basis functions

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Hermite Basis

- Specify curve by
  - Endpoint values
  - Endpoint tangents (derivatives)
- Parameter interval is arbitrary (most times)
  - Don't need to recompute basis functions
- These are ***cubic*** Hermite
  - Could do construction for any odd degree
  - $(d-1)/2$ derivatives at end points

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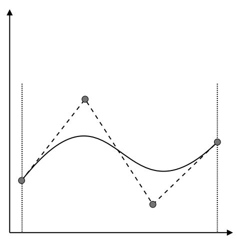
Cubic Bézier

- Similar to Hermite, but specify tangents indirectly

$$
\begin{aligned}
x_0 &= p_0 \\
x_1 &= p_3 \\
x_0' &= 3(p_1 - p_0) \\
x_1' &= 3(p_3 - p_2)
\end{aligned}
$$



Note: all the control points are points in space, no tangents.

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Cubic Bézier

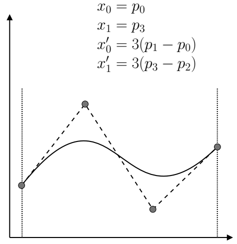
- Similar to Hermite, but specify tangents indirectly

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \mathbf{p}$$

$$\mathbf{c} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}} \mathbf{p}$$

$$\mathbf{c} = \boldsymbol{\beta}_Z \, \mathbf{p}$$

$$x_0 = p_0$$
$$x_1 = p_3$$
$$x_0' = 3(p_1 - p_0)$$
$$x_1' = 3(p_3 - p_2)$$



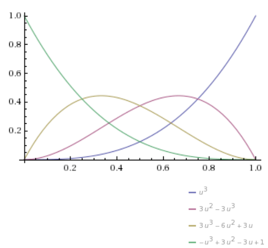
Cubic Bézier

Bézier basis functions

$$\mathbf{c} = \boldsymbol{\beta}_Z \, \mathbf{p} \qquad \mathbf{c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \mathbf{p}$$

$$x(u) = \mathcal{P}^3 \cdot \mathbf{c}$$

$$x(u) = \begin{bmatrix} 1 - 3u + 3u^2 - 1u^3 \\ 0 + 3u - 6u^2 + 3u^3 \\ 0 + 0u + 3u^2 - 3u^3 \\ 0 + 0u + 0u^2 + 1u^3 \end{bmatrix} \mathbf{p}$$



Changing Bases

- Power basis, Hermite, and Bézier all are still just cubic polynomials
  - The three basis sets all span the same space
  - Like different axes in
- Changing basis $\Re^{\cancel{3}}\ \Re^4$

$$\mathbf{c} = \boldsymbol{\beta}_{\mathrm{Z}}\,\mathbf{p}_{\mathrm{Z}}$$

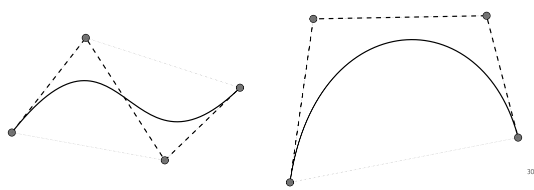
$$\mathbf{c} = \boldsymbol{\beta}_{\mathrm{H}}\,\mathbf{p}_{\mathrm{H}}$$

$$\mathbf{p}_{\mathrm{Z}} = \boldsymbol{\beta}_{\mathrm{Z}}^{-1}\,\boldsymbol{\beta}_{\mathrm{H}}\,\mathbf{p}_{\mathrm{H}}$$

Useful Properties of a Basis

- Convex Hull
  - All points on curve inside convex hull of control points
    - Bézier basis has convex hull property

$$\sum_i b_i(u) = 1 \qquad b_i(u) \geq 0 \qquad \forall u \in \Omega$$



Useful Properties of a Basis

- Invariance under class of transforms
  - Transforming curve is same as transforming control points
    - Bézier basis invariant for affine transforms
    - Bézier basis NOT invariant for perspective transforms
      - NURBS are though...

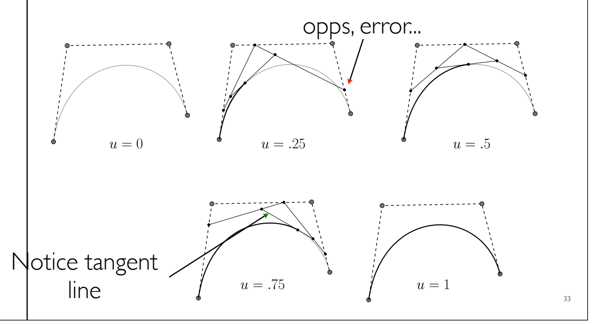
$$\boldsymbol{x}(u) = \sum_i \boldsymbol{p}_i b_i(u) \Leftrightarrow \boldsymbol{T}\boldsymbol{x}(u) = \sum_i (\boldsymbol{T}\boldsymbol{p}_i) b_i(u)$$

Useful Properties of a Basis

- Local support
  - Changing one control point has limited impact on entire curve
- Nice subdivision rules
- Orthogonality ($\int_\Omega b_i(u) b_j(u) \mathrm{d}u = \delta_{ij}$)
- Fast evaluation scheme
- Interpolation -vs- approximation

DeCasteljau Evaluation

- A geometric evaluation scheme for Bézier



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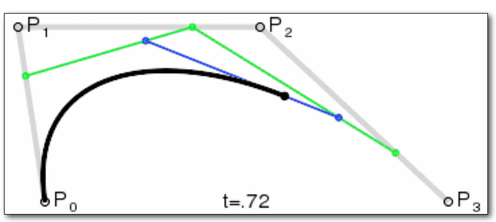
DeCasteljau Evaluation

Blue line is always tangent to curve.



From Wikipedia

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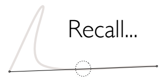
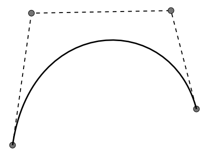
Adaptive Tessellation

- Midpoint test subdivision
- Possible problem
  - Simple solution if curve basis has ***convex hull*** property



If curve inside convex hull and the convex hull is nearly flat: curve is nearly flat and can be drawn as straight line



Better: draw convex hull

Works for Bézier because the ends are interpolated

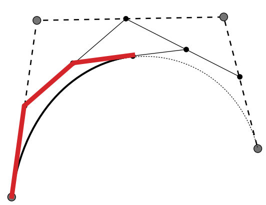
Bézier Subdivision

- Form control polygon for half of curve by evaluating at u=0.5

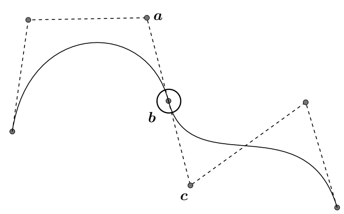
Repeated subdivision makes smaller/flatter segments

Also works for surfaces...

We'll extend this idea later on...



Joining



$\mathcal{C}^0 \Leftrightarrow \boldsymbol{b} = \boldsymbol{b}$

$\mathcal{C}^1 \Leftrightarrow \boldsymbol{b} - \boldsymbol{a} = \boldsymbol{c} - \boldsymbol{b}$

$\mathcal{G}^1 \Leftrightarrow \frac{\boldsymbol{b} - \boldsymbol{a}}{||\boldsymbol{b} - \boldsymbol{a}||} = \frac{\boldsymbol{c} - \boldsymbol{b}}{||\boldsymbol{c} - \boldsymbol{b}||}$

If you change ***a***, ***b***, or ***c*** you must change the others

But if you change ***a***, ***b***, or ***c*** you do not have to change beyond those three. *LOCAL SUPPORT*

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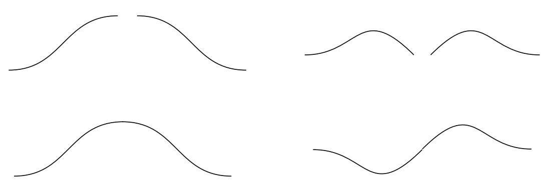
"Hump" Functions

- Constraints at joining can be built in to make new basis



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Tensor-Product Surfaces

- Surface is a curve swept through space
- Replace control points of curve with other curves

$$x(u,v) = \Sigma_i \, p_i \, b_i(u)$$
$$\Sigma_i \, q_i(v) \, b_i(u) \qquad q_i(v) = \Sigma_j \, p_{ji} \, b_j(v)$$

$$x(u,v) = \sum_{ij} p_{ij} b_i(u) b_j(v) \qquad b_{ij}(u,v) = b_i(u) b_j(v)$$

$$x(u,v) = \sum_{ij} p_{ij} b_{ij}(u,v)$$

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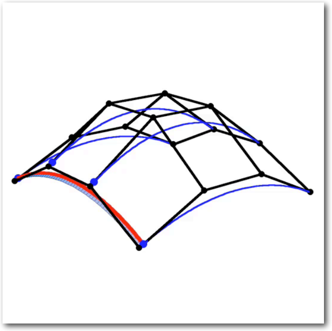
Tensor-Product Surfaces



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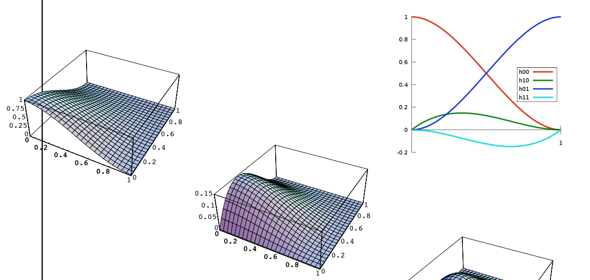
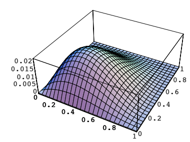
Hermite Surface Bases





Plus symmetries...

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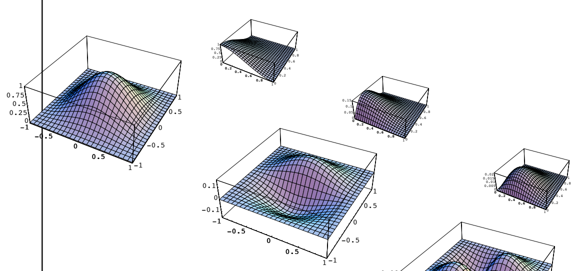
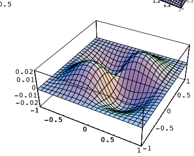
Hermite Surface Hump Functions





Plus symmetries...

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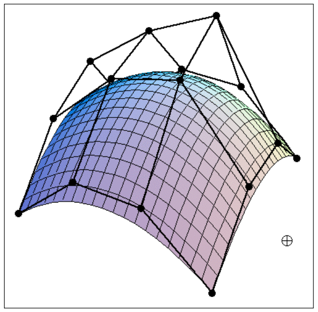
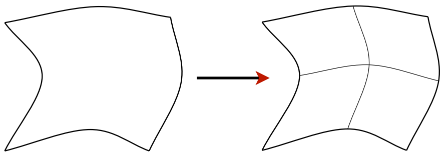
Bézier Surface Patch



Bezier surface and 4 x 4 array of control points

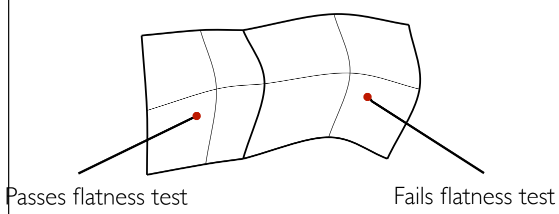
Adaptive Tessellation

- Given surface patch
  - If close to flat: draw it
  - Else subdivide 4 ways



Adaptive Tessellation

- Avoid cracking



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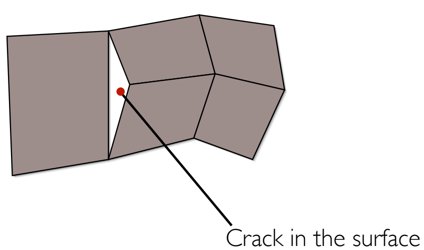
Adaptive Tessellation

- Avoid cracking



Cracks may be okay in some contexts...

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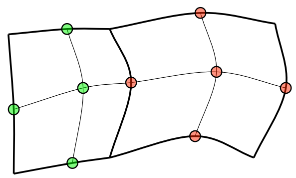
Adaptive Tessellation

- Avoid cracking



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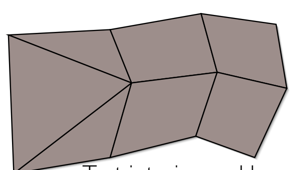
Adaptive Tessellation

- Avoid cracking



Test interior and boundary of patch
Split boundary based on boundary test
Table of polygon patterns
May wish to avoid "slivers"

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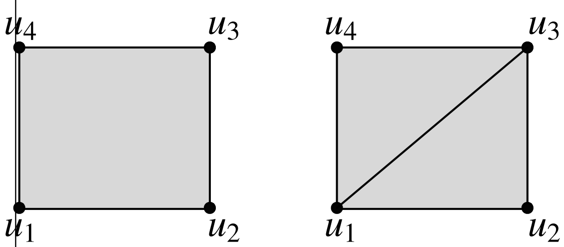
Adaptive Tessellation

- Triangle Based Method (no cracks)



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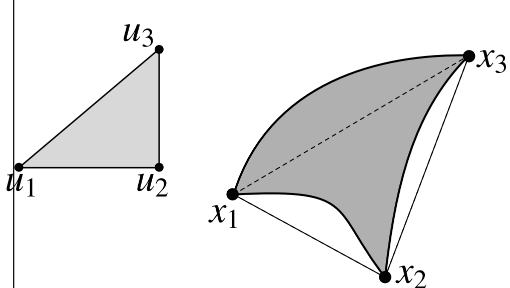
Adaptive Tessellation

- Triangle Based Method (no cracks)



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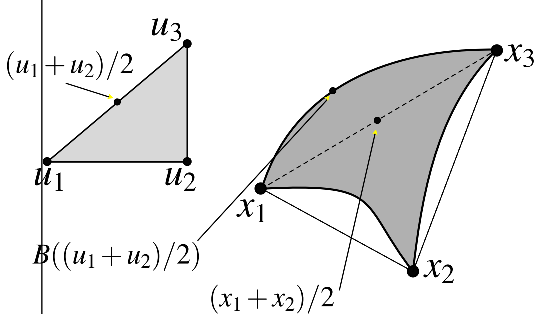
Adaptive Tessellation

- Triangle Based Method (no cracks)



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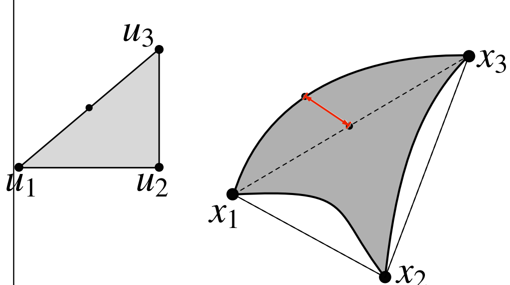
Adaptive Tessellation

- Triangle Based Method (no cracks)



$||B((u_1+u_2)/2)-(x_1+x_2)/2|| < \tau$?

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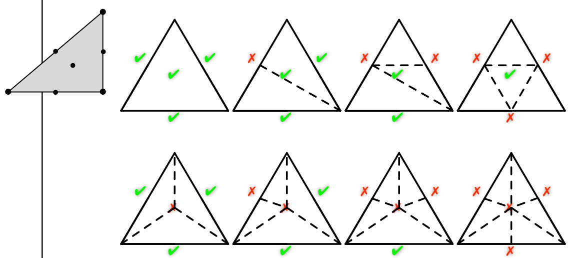
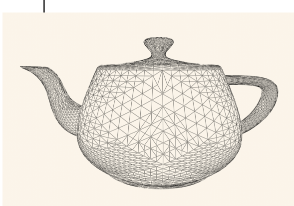
Adaptive Tessellation

- Triangle Based Method (no cracks)



Center test tends to generate slivers.
Often better to leave it out.

Adaptive Tessellation



Without center test

With center test

Yiding Jia, CS184 S08

Adaptive Tessellation

Second row shows typical error of swapping tests.

Yiding Jia, CS184 S08 -- I broke his code to make this example.

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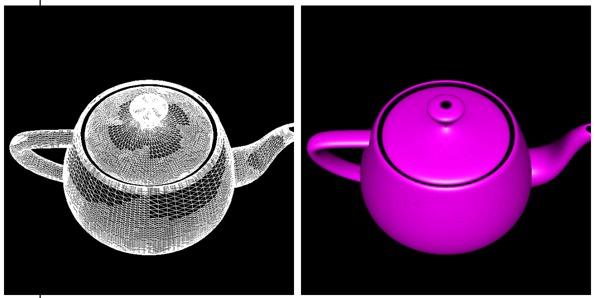
Adaptive Tessellation



Visible artifacts from cracks.

Apollo Ellis, CS184 S08

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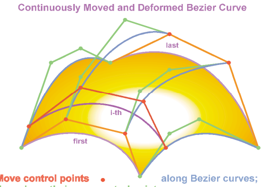
Bezier Surfaces. Smooth Operators.

```
# given the control points of a bezier curve
# and a parametric value, return the curve
# point and derivative
bezcurveinterp(curve, u)
    # first, split each of the three segments
    # to form two new ones AB and BC
    A = curve[0] * (1.0-u) + curve[1] * u
    B = curve[1] * (1.0-u) + curve[2] * u
    C = curve[2] * (1.0-u) + curve[3] * u

    # now, split AB and BC to form a new segment DE
    D = A * (1.0-u) + B * u
    E = B * (1.0-u) + C * u

    # finally, pick the right point on DE,
    # this is the point on the curve
    p = D * (1.0-u) + E * u

    # compute derivative also
    dPdu = 3 * (E - D)

    return p, dPdu
```

```
# given a control patch and (u,v) values, find
# the surface point and normal
bezpatchinterp(patch, u, v)
    # build control points for a Bezier curve in v
    vcurve[0] = bezcurveinterp(patch[0][0:3], u)
    vcurve[1] = bezcurveinterp(patch[1][0:3], u)
    vcurve[2] = bezcurveinterp(patch[2][0:3], u)
    vcurve[3] = bezcurveinterp(patch[3][0:3], u)

    # build control points for a Bezier curve in u
    ucurve[0] = bezcurveinterp(patch[0:3][0], v)
    ucurve[1] = bezcurveinterp(patch[0:3][1], v)
    ucurve[2] = bezcurveinterp(patch[0:3][2], v)
    ucurve[3] = bezcurveinterp(patch[0:3][3], v)

    # evaluate surface and derivative for u and v
    p, dPdv = bezcurveinterp(vcurve, v)
    p, dPdu = bezcurveinterp(ucurve, u)

    # take cross product of partials to find normal
    n = cross(dPdu, dPdv)
    n = n / length(n)

    return p, n
```

Bicubic Bezier Patch

Continuously Moved and Deformed Bezier Curve



Move control points along Bezier curves;
these have their own control points ;
leading to a total of 16 control points for the cubic case.

```
# given a patch, perform uniform subdivision
subdividepatch(patch, step)
    # compute how many subdivisions there
    # are for this step size
    numdiv = ((1 + epsilon) / step)

    # for each parametric value of u
    for (iu = 0 to numdiv)
        u = iu * step

        # for each parametric value of v
        for (iv = 0 to numdiv)
            v = iv * step

            # evaluate surface
            p, n = bezpatchinterp(patch, u, v)
            savesurfacepointandnormal(p,n)
```

Split?

e3	e2	e1	
0	0	0	output as is
0	0	1	
0	1	0	
1	0	0	
0	1	1	
1	1	0	
1	0	1	
1	1	1