Subdivision

- Start with:
  - Given control points for a curve or surface, find new control points for a sub-section of curve/surface.
- Key extension to basic idea:
  - Generalize to non-regular surfaces.
Consider NURBS Surface

Control mesh dictates feature size.

Coarse mesh

Displaced CV

Large bump

Fine mesh

Displaced CV

Small bump

Excessive detail
Tensor Product Surface Refinement

Refinement must be constant across \( u \) or \( v \) directions.

Bézier Subdivision

\[
\begin{align*}
\mathbf{x}(u) &= \sum_b b_i(u) \mathbf{p}_i \nonumber \\
\mathbf{x}(u) &= [1, u, u^2, u^3] \mathbf{b}_z \mathbf{P} 
\end{align*}
\]

\( \mathbf{b}_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \)
Bézier Subdivision

\[ x(u) = [1, u, u^2, u^3] \beta_z P \]

\[ \beta_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \]

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Bézier Subdivision

\[ x(u) = [1, u, u^2, u^3] \beta_z P \]

\[ s_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} \]

\[ x(u) = [1, u, u^2, u^3] \beta_z S_1 \beta_z P \]

\[ x(u) = [1, u, u^2, u^3] \beta_z H_{z_1} P \]

Bézier Subdivision

\[ x(u) = [1, u, u^2, u^3] \beta_z H_{z_1} P \]

\[ x(u) = [1, u, u^2, u^3] \beta_z P_1 \]

\[ H_{z_1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ P_1 = H_{z_1} P \]
Bézier Subdivision

\[ \mathbf{x}(u) = [1, u, u^2, u^3] \beta_z \mathbf{P}_2 \]

\[ \mathbf{P}_2 = \mathbf{H}_{Z2} \mathbf{P} \]

\[ \mathbf{H}_{Z2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ \mathbf{S}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Bézier Subdivision

\[ \mathbf{P}_2 = \mathbf{H}_{Z2} \mathbf{P} \]

\[ \mathbf{x}(u, v) = [1, u, u^2, u^3] \beta_z \mathbf{P}_2 \beta_z^T [1, v, v^2, v^3]^T \]

4 x 4 matrix of control points
Bézier Subdivision

\[
x(u, v) = \begin{bmatrix} 1, u, u^2, u^3 \end{bmatrix}^T \beta_2 \mathbf{P}_Z \beta_2^T \begin{bmatrix} 1, v, v^2, v^3 \end{bmatrix}^T
\]

\[\mathbf{P}_{21} = \mathbf{H}_{Z2} \mathbf{P} \mathbf{H}_{Z1}^T\]

Regular B-Spline Subdivision

Orthographic top-down

3D Perspective view
Regular B-Spline Subdivision

\[ x(u, v) = [1, u, u^2, u^3] \beta_B \mathbf{P} \beta_B^T [1, v, v^2, v^3]^T \]

\[ \mathbf{P}_{11} = H_{B1} \mathbf{P} H_{B1}^T \]
Regular B-Spline Subdivision

\[ P_{11} = H_{B1} P H_{B1}^T \]

In this parametric view these knot points are collocated.
The 3D control points are not.
Regular B-Spline Subdivision

\[ P_{11} = H_{B1} P H_{B1}^T \]
\[ P_{12} = H_{B1} P H_{B2}^T \]
\[ P_{22} = H_{B2} P H_{B2}^T \]
\[ P_{21} = H_{B2} P H_{B1}^T \]
Regular B-Spline Subdivision

\[ \mathbf{P}^{i+1} = \mathbf{H} \mathbf{P}^i \]

Length 16 vector of coarse CPs

Length 25 vector of fine CPs

25 x 16 subdivision matrix

Inspection would reveal a pattern:

• Face points
• Edge points
• Vertex points
Regular B-Spline Subdivision

Recall that control mesh approaches surface.
Regular B-Spline Subdivision

- Limit of subdivision is the surface

Irregular B-Spline Subdivision

- Catmull-Clark Subdivision
  - Generalizes regular B-Spline subdivision

An irregular patch
Non-quad face
Extraordinary vertex
### Irregular B-Spline Subdivision

- Catmull-Clark Subdivision
  - Generalizes regular B-Spline subdivision
  - Rules reduce to regular for ordinary vertices/faces

\[
f = \text{average of surrounding vertices}
\]
\[
e = \frac{f_1 + f_2 + v_1 + v_2}{4}
\]
\[
v = \frac{f + 2m}{n} + \frac{p(n-3)}{n}
\]

- \(\bar{m}\) = average of adjacent midpoints
- \(\bar{f}\) = average of adjacent face points
- \(n\) = valence of vertex

### Catmull-Clark Subdivision

- Diagram showing a subdivision process with vertices and midpoints.
Catmull-Clark Subdivision

- In “ordinary” regions:
  - Surface is fully $C^2$ everywhere except extraordinary points
  - Fast evaluation by matrix exponentiation
- At extraordinary points:
  - Surface is at least $C^1$
  - Curvature is Lipschitz continuous at extraordinary points

Continuity of Catmull-Clark