CS-184: Computer Graphics

Lecture #13: Natural Splines, B-Splines, and NURBS

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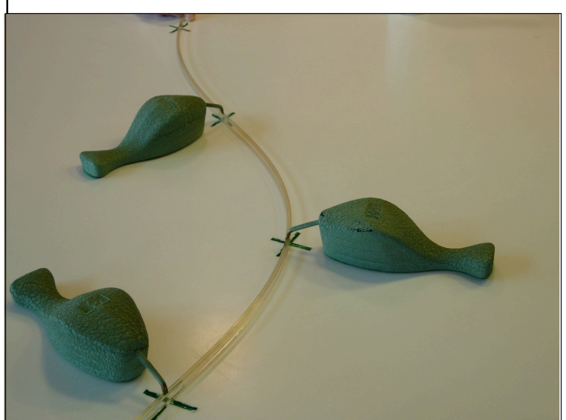
Natural Splines

• Draw a "smooth" line through several points



A real draftsman's spline.

Image from Carl de Boor's webpage.

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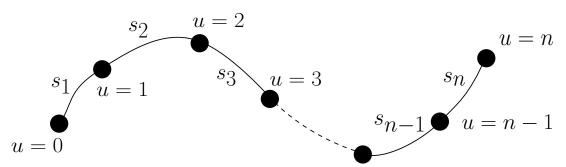
Natural Cubic Splines

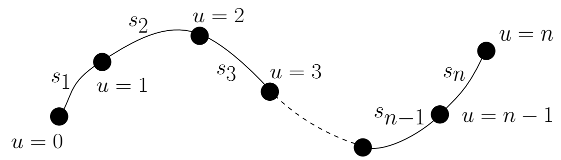
- Given $n+1$ points
  - Generate a curve with n segments
  - Curves passes through points
  - Curve is C^2 continuous
- Use cubics because lower order is better...

Natural Cubic Splines



$$\mathbf{x}(u) = \begin{cases} \mathbf{s}_1(u) & \text{if } 0 \le u < 1 \\ \mathbf{s}_2(u-1) & \text{if } 1 \le u < 2 \\ \mathbf{s}_3(u-2) & \text{if } 2 \le u < 3 \\ & \vdots \\ \mathbf{s}_n(u-(n-1)) & \text{if } n-1 \le u \le n \end{cases}$$

Natural Cubic Splines



$s_i(0) = p_{i-1} \quad i = 1 \ldots n$ ← n constraints
$s_i(1) = p_i \quad i = 1 \ldots n$ ← n constraints

$s_i'(1) = s_{i+1}'(0) \quad i = 1 \ldots n-1$ ← n-1 constraints
$s_i''(1) = s_{i+1}''(0) \quad i = 1 \ldots n-1$ ← n-1 constraints

$s_1''(0) = s_n''(1) = 0$ ←2 constraints

Total 4n constraints

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Natural Cubic Splines

- Interpolate data points
- No convex hull property
- Non-local support
  - Consider matrix structure...
- C^2 using cubic polynomials

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B-Splines

- Goal: C^2 cubic curves with local support
  - Give up interpolation
  - Get convex hull property
- Build basis by designing “hump” functions

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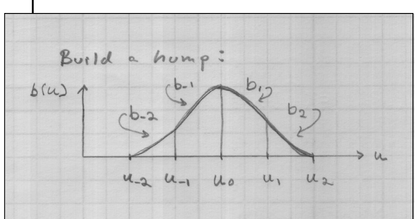
B-Splines



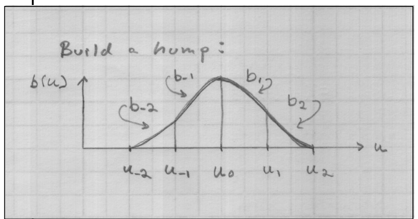
$$\mathbf{b}(u) = \begin{cases} \mathbf{b_{-2}}(u) & \text{if } u_{-2} \leq u < u_{-1} \\ \mathbf{b_{-1}}(u) & \text{if } u_{-1} \leq u < u_0 \\ \mathbf{b_{+1}}(u) & \text{if } u_0 \leq u < u_{+1} \\ \mathbf{b_{+2}}(u) & \text{if } u_{+1} \leq u \leq u_{+2} \end{cases}$$

$b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0$ ←3 constraints
$b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) = 0$ ←3 constraints

$b_{-2}(u_{-1}) = b_{-1}(u_{-1})$
$b_{-1}(u_0) = b_{+1}(u_0)$
$b_{+1}(u_{+1}) = b_{+2}(u_{+1})$
← [Repeat for b' and b''
3×3=9 constraints

Total 15 constraints ...... need one more

B-Splines



$$\mathbf{b}(u) = \begin{cases} \mathbf{b_{-2}}(u) & \text{if } u_{-2} \leq u < u_{-1} \\ \mathbf{b_{-1}}(u) & \text{if } u_{-1} \leq u < u_0 \\ \mathbf{b_{+1}}(u) & \text{if } u_0 \leq u < u_{+1} \\ \mathbf{b_{+2}}(u) & \text{if } u_{+1} \leq u \leq u_{+2} \end{cases}$$

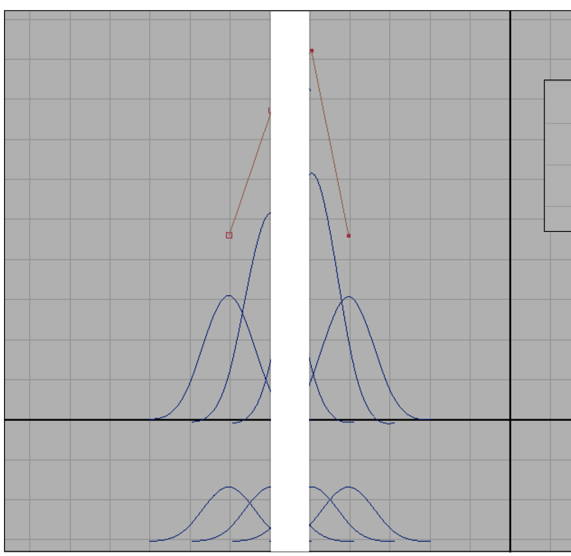
$b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0$ ←3 constraints
$b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) = 0$ ←3 constraints

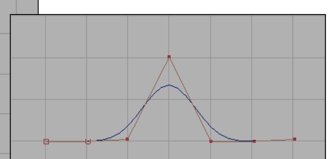
$b_{-2}(u_{-1}) = b_{-1}(u_{-1})$
$b_{-1}(u_0) = b_{+1}(u_0)$
$b_{+1}(u_{+1}) = b_{+2}(u_{+1})$
← [Repeat for b' and b''
3×3=9 constraints

$b_{-2}(u_{-2}) + b_{-1}(u_{-1}) + b_{+1}(u_0) + b_{+2}(u_{+1}) = 1$ ←1 constraint (convex hull)

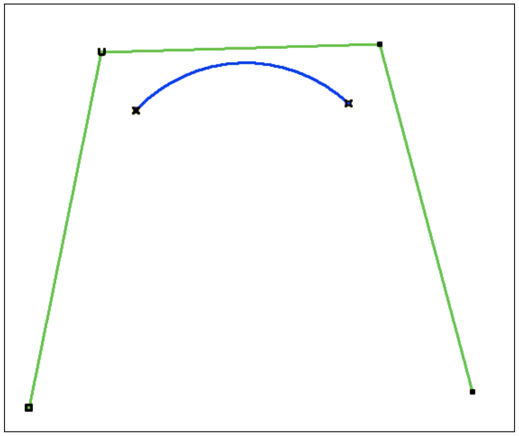
Total 16 constraints

B-Splines





B-Splines



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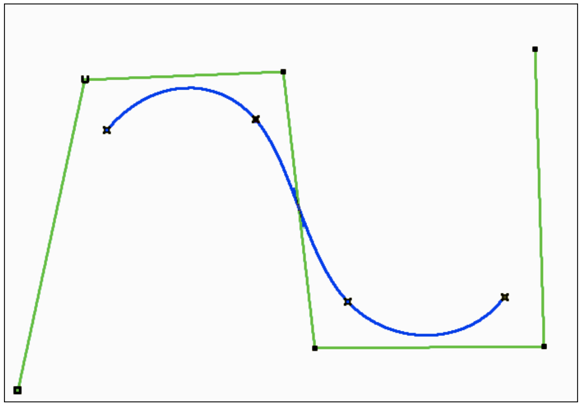
B-Splines



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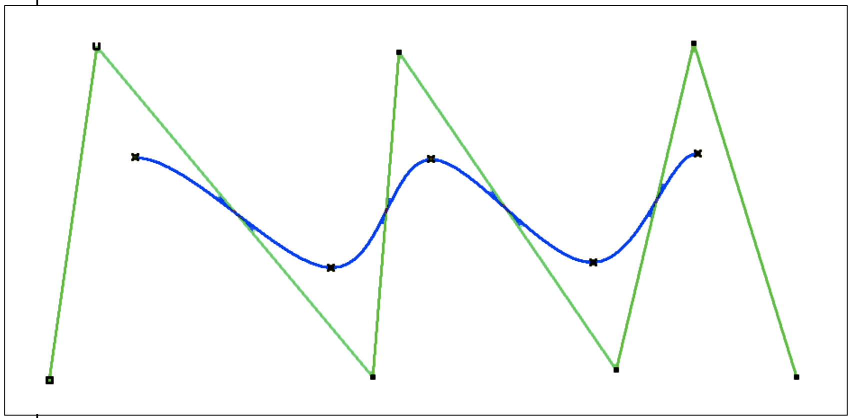
B-Splines



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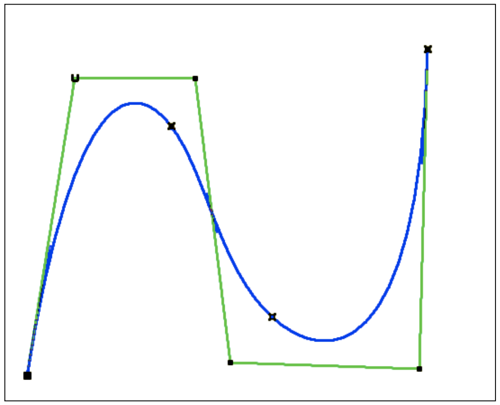
B-Splines



Example with end knots repeated

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B-Splines

- Build a curve w/ overlapping bumps
- Continuity
  - Inside bumps C^2
  - Bumps "fade out" with C^2 continuity
- Boundaries
  - Circular
  - Repeat end points
  - Extra end points

B-Splines

- Notation
  - The basis functions are the $b_i(u)$
  - "Hump" functions are the concatenated function
    - Sometimes the humps are called basis... can be confusing
  - The u_i are the knot locations
  - The weights on the hump/basis functions are control points

B-Splines

- Similar construction method can give higher continuity with higher degree polynomials
- Repeating knots drops continuity
  - Limit as knots approach each other
- Still cubics, so conversion to other cubic basis is just a matrix multiplication

B-Splines

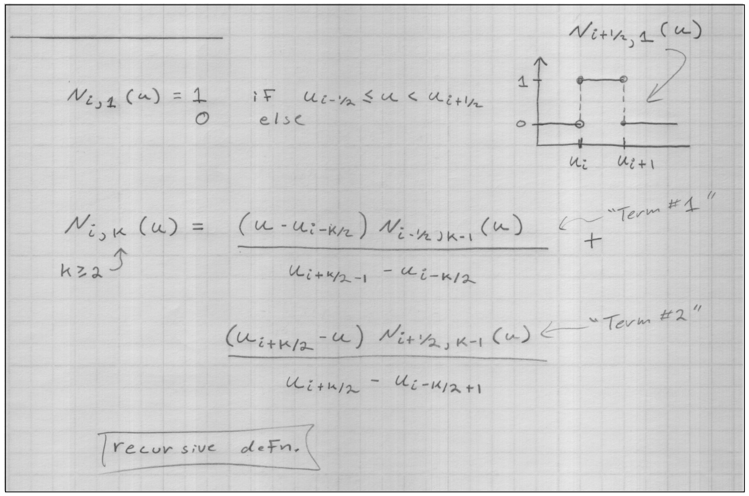
- Geometric construction
  - Due to Cox and de Boor
  - My own notation, beware if you compare w/ text
- Let hump centered on u_i be $N_{i,4}(u)$

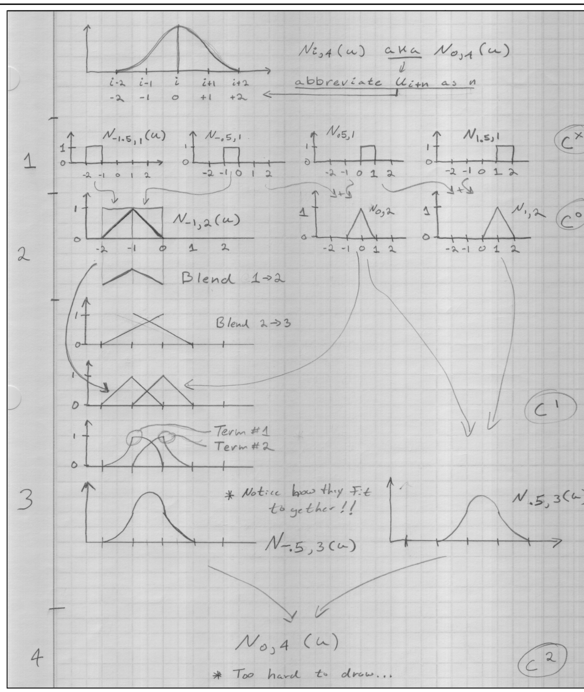
Cubic is order 4

$N_{i,k}(u)$ Is order k hump, centered at u_i

Note: i is integer if k is even
else $(i+1/2)$ is integer

B-Splines


$N_{i,1}(u) = 1$ if $u_{i-1/2} \le u < u_{i+1/2}$
0 else
$N_{i+1/2,1}(u)$
u_i u_{i+1}
$N_{i,k}(u) = \frac{(u - u_{i-k/2}) N_{i-1/2,k-1}(u)}{u_{i+k/2-1} - u_{i-k/2}}$ + $\frac{(u_{i+k/2} - u) N_{i+1/2,k-1}(u)}{u_{i+k/2} - u_{i-k/2+1}}$
"Term #1"
"Term #2"
$k \ge 2$
recursive defn.


$N_{i,4}(u)$ aka $N_{0,4}(u)$
abbreviate u_{i+n} as n
C^∞
C^0
Blend 1→2
Blend 2→3
Term #1
Term #2
C^1
* Notice how they fit together!!
$N_{-.5,3}(u)$
$N_{.5,3}(u)$
$N_{0,4}(u)$
* Too hard to draw...
C^2

NURBS

- **N**on**u**niform **R**ational **B**-**S**plines
  - Basically B-Splines using homogeneous coordinates
  - Transform under perspective projection
  - A bit of extra control

NURBS

$$\mathbf{p}_i = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \\ p_{iw} \end{bmatrix}$$

$$\mathbf{x}(u) = \frac{\sum_i \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \end{bmatrix} N_i(u)}{\sum_i p_{iw} N_i(u)}$$

- Non-linear in the control points
- The p_{iw} are sometimes called "weights"