EECS 225A Statistical Signal Processing Lecture 1 - 1/21/2020

Lecture 1: Reviewing DTFT and z-transform

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In this lecture we review some basic concepts of signal processing, including the Discrete-Time Fourier Transform (DTFT) and z-transform.

1 Discrete-Time Fourier Transform (DTFT) [1, Chapter 3.4]

We define the *Discrete-Time Fourier Transform* (DTFT) of the signal as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}.$$

In the literature, sometimes we also write it as $X(e^{j\omega})$ since it in fact only depends on $e^{j\omega}$, and that $X(\omega)$ is a periodic function with period 2π. We usually only specify $X(\omega)$ for $\omega \in (-\pi, \pi]$.

The inverse DTFT is given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

A natural question is, for what family of signals $x[n]$ does the DTFT exist? The following three cases are of particular significance:

- ℓ^1 class: $\sum_{k=-\infty}^{\infty} |x[k]| < \infty$, i.e. series that converge absolutely. For this family, $X(\omega)$ is well defined for every ω, the convergence is uniform over ω, and $X(\omega)$ is a continuous function of ω. [1]

- ℓ^2 class: $\sum_{k=-\infty}^{\infty} |x[k]|^2 < \infty$, i.e. square-summable sequences. This class is bigger than ℓ^1 and corresponds to the Hilbert space theory of Fourier transform, and the corresponding DTFT "converges in a mean-squared sense":

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |X_N(\omega) - X(\omega)|^2 d\omega = 0,$$

  where $X_N(\omega)$ refers to the DTFT of $x[k]$ truncated to between $k = -N$ and N, which is a finite sequence so $X_N(\omega)$ is well defined. For ℓ^2 family, $X(\omega)$ is only defined "almost everywhere". Let us consider the important example of a (discrete) sinc function and its DTFT, where $0 < \omega_0 < \pi$:

$$x[k] = \frac{\sin(\omega_0 k)}{\pi k} \quad \longleftrightarrow \quad X(\omega) = \begin{cases} 1 & \text{if } |\omega| < \omega_0 \\ 0 & \text{else} \end{cases}$$

  The sinc function belongs to the ℓ^2 class but not ℓ^1. Compare $\sum_k \frac{1}{k}$ with $\sum_k \frac{1}{k^2}$ for intuition about convergence. We can see that its DTFT is not continuous in ω. In fact, $X_N(\omega)$ does not converge for $\omega = \omega_0$.

[1] Absolute convergence implies uniform convergence of sequence of functions $X_N(\omega) \triangleq \sum_{n=-N}^{N} x[n]e^{-j\omega n}$ to X. Looking at the DTFT as a function defined on a compact domain such as $[-\pi, \pi]$, the uniform convergence and the continuity of X_N for each N implies that X is continuous.

- Tempered distributions: it is bigger than ℓ^1 and ℓ^2, but requires the theory of generalized functions, or distributions. We will not discuss the details here, but in signal processing literature we usually use the Dirac delta function to deal with this case. For example, for constant signal: $x[k] \equiv 1$, which does not belong to ℓ^1 or ℓ^2, we have the DTFT pair

$$x[k] = 1 \quad \longleftrightarrow \quad X(\omega) = 2\pi\delta(\omega)$$

Applying the IDTFT formula, $x[k] = \int_{-\pi}^{\pi} \delta(\omega)d\omega = 1$, as desired, by the definition of the δ function.

The IDTFT formula holds for all signals in these three classes.

1.1 Properties of DTFT

The DTFT pair $x[n] \longleftrightarrow X(\omega)$ has the following properties:

1. time delay:

$$x[n-k] \longleftrightarrow e^{-j\omega k}X(\omega) \tag{1}$$

2. time reversal:

$$x[-n] \longleftrightarrow X(-\omega) \tag{2}$$

3. conjugate:

$$x^*[n] \longleftrightarrow X^*(-\omega) \tag{3}$$

4. convolution in time:

$$x_1[n] * x_2[n] \triangleq \sum_{l=-\infty}^{\infty} x_1[l]x_2[n-l] \longleftrightarrow X_1(\omega)X_2(\omega) \tag{4}$$

5. deterministic crosscorrelation:

$$c[n] = \sum_{k\in\mathbb{Z}} x[k]y^*[k-n] \Longleftrightarrow C(\omega) = X(\omega)Y^*(\omega). \tag{5}$$

2 z-transform [1, Chapter 3.5]

The z-transform generalizes the DTFT by replacing the complex number $e^{j\omega}$ with general complex number $z \in \mathbb{C}$.

2.1 Definition and Convergence

A discrete-time signal is denoted by

$$x[n]\ ,\ n \in \mathbb{Z} \tag{6}$$

The z-transform of $x[n]$ is defined as

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}\ ,\ z \in \mathbb{C} \tag{7}$$

The series in (7) is called a Laurent series. Complex analysis theory tells us that there exists a unique inner radius r and outer radius R such that

1. The Laurent series converges *absolutely* on the open annulus $A \triangleq \{z \mid r < |z| < R\}$. To say the series converges, we mean that both the positive degree power series and the negative degree power series absolutely converge, which is equivalent to

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty. \tag{8}$$

   Furthermore, this convergence will be uniform on compact sets. Finally, the convergence series defines a holomorphic function $f(z)$ on the open annulus.

2. Outside the annulus, the Laurent series diverges. That is, at each point of the exterior of A, the positive degree power series or the negative degree power series diverges.

3. On the boundary of the annulus, one cannot make a general statement, except to say that there is at least one point on the inner boundary and one point on the outer boundary such that $f(z)$ cannot be holomorphically continued to those points.

It is possible that $r = 0$ or $R = \infty$. It is also not true that $r < R$ in general. We can compute the radii r, R as follows:

$$r = \limsup_{n\to\infty} |x[n]|^{1/n} \tag{9}$$

$$\frac{1}{R} = \limsup_{n\to\infty} |x[-n]|^{1/n}. \tag{10}$$

In signal processing, we define the *region of convergence (ROC)* as the set $\{z \mid r < |z| < R\}$ since we only care about absolute convergence. By convention, the ROC concept is extended to $|z| = \infty$ by including $|z| = \infty$ in the ROC when $x[n] = 0$ for all $n < 0$ and excluding it otherwise. Similarly, $z = 0$ is in the ROC when $x[n] = 0$ for all $n > 0$ and not in the ROC otherwise.

2.2 ROC is needed to describe the signal

It is essential to specify the ROC along with the z-transform $X(z)$ to uniquely describe a discrete-time signal. This is because 2 different signals $x_1[n]$ and $x_2[n]$ can have the same z-transform $X(z)$ and can only be distinguished by their different regions of convergence, as highlighted in the following 2 examples.

Note: The Heaviside step function is defined by

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

2.2.1 Example 1

Consider the signal

$$x_1[n] = (\frac{1}{2})^n u[n] \tag{12}$$

The z-transform is given by

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} \tag{13}$$

$$= \sum_{n=0}^{\infty} (\frac{1}{2z})^n \tag{14}$$

$$= \frac{1}{1 - \frac{1}{2z}} \tag{15}$$

In (14) we use the relation $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ which converges iff $|a| < 1$. Hence the ROC for the z-transform is

$$|\frac{1}{2z}| < 1 \Rightarrow |z| > \frac{1}{2} \tag{16}$$

2.2.2 Example 2

Consider the signal

$$x_2[n] = -(\frac{1}{2})^n u[-n-1] \tag{17}$$

The z-transform is given by

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} = -\sum_{n=-\infty}^{-1} (\frac{1}{2})^n z^{-n} \tag{18}$$

$$= -\sum_{m=1}^{\infty} (2z)^m \tag{19}$$

$$= -\frac{2z}{1-2z} = \frac{1}{1 - \frac{1}{2z}} \tag{20}$$

The ROC for the z-transform is

$$|2z| < 1 \Rightarrow |z| < \frac{1}{2} \tag{21}$$

From (15) and (20) we see that the z-transforms are equal for the 2 signals $x_1[n]$ and $x_2[n]$, but the ROCs in (16) and (21) are different.

2.3 Causality

We say a discrete-time signal $x[n]$ is causal if and only if $x[n] = 0$ for $n < 0$, hence we know $x[n]$ is causal if and only if the ROC includes ∞.

From the ROCs in the two examples above, we can see that only the first ROC (16) includes ∞, so $x_1[n]$ is causal while $x_2[n]$ is noncausal, which is also obvious from the time domain representations.

2.4 BIBO stability

BIBO stands for bounded input, bounded output. By Holder's inequality,

$$\left| \sum_{n=-\infty}^{\infty} x[n]y[n] \right| \leq \sup_n |y[n]| \sum_{n=-\infty}^{\infty} |x[n]| \tag{22}$$

Since the dual norm of ℓ^∞-norm is ℓ^1 norm, the left hand side in (22) is bounded for all bounded (ℓ^∞-norm bounded) inputs y iff $\sum_{n=-\infty}^{\infty} |x[n]| < \infty \iff$ ROC contains the unit circle $|z| = 1$ (from absolute convergence criteria)

Once again, from the ROCs in the two preceding examples, we can see that only the first ROC (16) includes $|z| = 1$, so $x_1[n]$ is BIBO stable while $x_2[n]$ is BIBO unstable, which can be inferred from the time domain representations as well.

2.5 Properties of z-transform

The z-transform pair $x[n] \longleftrightarrow X(z)$ has the following properties:

1. time delay:

$$x[n-k] \longleftrightarrow z^{-k} X(z) \tag{23}$$

2. time reversal:

$$x[-n] \longleftrightarrow X(z^{-1}) \tag{24}$$

3. conjugate:

$$x^*[n] \longleftrightarrow X^*(z^*) \tag{25}$$

4. convolution in time:

$$x_1[n] * x_2[n] \triangleq \sum_{l=-\infty}^{\infty} x_1[l] x_2[n-l] \longleftrightarrow X_1(z) X_2(z) \tag{26}$$

5. deterministic crosscorrelation:

$$c[n] = \sum_{k \in \mathbb{Z}} x[k] y^*[k-n] \Longleftrightarrow C(z) = X(z) Y_*(z^{-1}), \tag{27}$$

   where $Y_*(z) \triangleq Y^*(z^*)$.

References

[1] M. Vetterli, J. Kovačević, and V. K. Goyal, *Foundations of signal processing.* Cambridge University Press, 2014.