Easily Computable Optimum Grasps in 2-D and 3-D

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Abstract

We consider the problem of finding optimum force closure grasps of two and three-dimensional objects. Our focus is on grasps which are useful in practice, namely grasps with a small number of fingers, with friction at the contacts. Assuming frictional contact and rounded finger tips—very mild assumptions in practice—we give new upper (and lower) bounds on the number of fingers necessary to achieve force closure grasps of 2-D and 3-D objects. We develop an optimality criterion based on the notion of decoupled wrenches, and use this criterion to derive optimum two and three finger grasps of 2-D objects, and optimum three finger grasps for 3-D objects. We present a simple $O(n)$ algorithm for computing these optimum grasps for convex polygons, a $O(n \log n)$ algorithm for non-convex polygons, and an $O(n^3)$ algorithm for polyhedra. In studying these optimum grasps, we derive several interesting theoretical results concerning grasp geometry.

1 Introduction

The theory of what constitutes a force closure grasp of an object is well understood, and much research has been devoted to constructing these grasps for various types of objects. Defining and constructing *optimum* force closure grasps is not as straightforward, however. Part of the problem stems from the fact that grasp optimality is dependent on what the grasp is expected to do, that is how it is to manipulate the grasped object or what types of forces and moments it is to resist. For example, a grasp which is better than another at resisting the gravitational forces on the object may not be as good at resisting moments in certain directions. While it is possible to base a grasp quality measure on how well a grasp can resist some generalized wrench, the non-comparability of forces and moments leads to problems with this scheme.

In this paper, we take a practical tack toward solving the optimum grasp problem. We are concerned with friction grasps of objects, using a small number of fingers, as this situation is the most typical in grasping applications. The accepted wisdom is that three (hard) fingers are required for a force closure grasp of a 2-D object, and four fingers are required to grasp a 3-D object [MNP90]. However, using a realistic model (rounded fingers and static friction at the contacts) we show that **any** 2-D object can be grasped with **two** fingers, and **any** 3-D object with **three** fingers.

We acknowledge that our optimality criteria are not the only ones, and other criteria could lead to different optimum grasps. Nonetheless, our criteria lead to very simple, easily characterized grasps, which seem intuitively to be good grasps.

1.1 Related work

The force closure problem was introduced into the robotics literature by Lakshminarayana, who established bounds on the minimum number of fingers needed to achieve force closure of rigid bodies [Lak78]. Mishra, Schwartz, and Sharir derived theoretical bounds on the minimum number of fingers required to hold bodies at equilibrium, with no friction; they also give linear time algorithms to synthesize such grasps for polygonal and polyhedral objects [MTS87]. Li and Sastry developed various "task-oriented" grasp optimality criteria which took into account the particular types of manipulations to be performed [LS87]. Markenscoff and Papadimitriou formulated and found optimum grasps of polygons, based on minimizing the applied finger forces necessary to balance some subset of full wrench space (e.g. balancing the object's weight) [MP89]. Our method is similar, although we consider a larger subset of wrench space, and also obtain results for three-dimensional objects. With Ni, they also proved that under very general conditions, force closure friction grasps of 2-D and 3-D objects can be obtained with three and four fingers, respectively.

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[MNP90]. We lower each of these bounds by one finger through a smoothness assumption that is borne out in practice because of rounded fingertips. Ponce, et. al. gave an algorithm to compute maximal grasp regions for three and four-finger polyhedral grasps [PSBM93]; Our algorithm assumes three rounded fingers, but is of lower complexity. Ferrari and Canny also studied the problem of optimum force closure grasps, however, the problem of the non-comparability of forces and moments is not completely addressed [FC92]. Park and Starr studied and developed algorithms for optimum three finger grasps of planar objects, based on a heuristic optimality function [PS92].

2 Formalizing grasp optimality

A grasp is applied to an object to hold it in equilibrium, or to manipulate or accelerate it in some manner. In either case, a grasp's quality somehow boils down to how wide a range of forces and moments can be imparted to the object, while keeping the finger forces within some bound. Henceforth, it is assumed the task is to hold the object in equilibrium, while it is under the influence of some other forces and moments which combine to form what will be termed the *external* wrench (roughly speaking, a wrench is combination of a force and a moment, $\vec{w} = (\vec{f}, \vec{m})^T$). We assume that the finger forces are applied in a direction normal to the object surface, so specifying the point on the object where the finger is applied also specifies the direction of the force. We are only considering force closure grasps, so any external wrench can be balanced by choosing finger forces appropriately.

Let $\mathcal{W}$ denote the space of all wrenches, let O be the surface of an object, viewed as a set of points, and consider the case of n fingers. We specify a grasp g by specifying the placement of the n fingers on the object, hence if $\mathcal{G}$ is the space of all grasps, $\mathcal{G}$ is isomorphic to O^n. Associated with each grasp is a grasp map defined as follows:

Definition 1 *For $\vec{p} = (p_1, \ldots, p_n) \in O^n$, we define the* **grasp map** $g_{\vec{p}} : \overline{\mathbb{R}_+}^n \to \mathcal{W}$ *by*

$$g_{\vec{p}}(f_1, \ldots, f_n) = \vec{w},$$

where $\vec{w}$ is the resultant wrench acting on the object when finger i applies a force of magnitude f_i along the inward normal at point p_i (for $i = 1, \ldots, n$). $g_{\vec{p}}$ is not injective; $g_{\vec{p}}^{-1}(\vec{w})$ is the set of all tuples of finger forces which impart the wrench $\vec{w}$ on the object. Henceforth, it is assumed that every grasp and grasp map correspond to some particular placement of fingers on the object, and we drop the subscript $\vec{p}$, writing simply $g(\vec{f})$ or $g^{-1}(\vec{w})$.

Geometric hand constraints may reduce $\mathcal{G}$ to a subset of O^n; we ignore these constraints and assume that each finger may be placed anywhere on the object.

For an externally applied wrench $\vec{w}$, $g^{-1}(-\vec{w})$ is a set of tuples of finger forces which can resist $\vec{w}$. A reasonable measure of how well a grasp g resists a particular wrench $\vec{w}$ is

$$\max_{\vec{f} \in g^{-1}(-\vec{w})} \frac{||\vec{w}||_w}{||\vec{f}||_f}.$$

The ratio of external wrench magnitude to finger force magnitude is maximized, since it is desirable to resist a wrench with finger forces as small as possible. However, a force closure grasp must resist all possible external wrenches, and since an adversary will choose the hardest wrench for a particular grasp to resist, the quality measure on the space of grasps must be:

$$Q(g) = \min_{\vec{w} \in \mathcal{W}} \left(\max_{\vec{f} \in g^{-1}(-\vec{w})} \frac{||\vec{w}||_w}{||\vec{f}||_f} \right).$$

This expression can be simplified by noting that both the force and moment components of the wrench applied by the fingers scale linearly with the magnitude of the finger forces, and so $g^{-1}(\lambda \vec{w}) = \lambda g^{-1}(\vec{w})$ for any $\lambda \geq 0$. Thus it is no loss of generality to restrict the minimization to the set of all unit wrenches in $\mathcal{W}$. If we call this set $\mathcal{W}_u$, we obtain

$$Q(g) = \min_{\vec{w} \in \mathcal{W}_u} \left(\max_{\vec{f} \in g^{-1}(-\vec{w})} \frac{1}{||\vec{f}||_f} \right). \tag{1}$$

We have not yet defined the norm $||\cdot||_w$ on wrench space nor the norm $||\cdot||_f$ on finger force space. Defining the norm on tuples of finger forces is straightforward; reasonable choices are an L_1 norm which minimizes the sum of the finger forces, or an L_∞ norm which minimizes the maximum finger force. However, defining a norm on wrench space is problematic, since forces and moments don't even have the same units. This problem can be solved by developing an optimality criterion based on *decoupled wrenches*. The basic idea is to consider how well a grasp resists some subset of wrenches from wrench space. In general many grasps may resist this category of wrenches equally well, and so these grasps are then evaluated on a different subset of wrenches to determine the best grasp overall. To avoid the problem of non-comparability of forces and moments, the subsets chosen from wrench space will consist of pure forces (through some distinguished point), or pure moments. Hence it will never be necessary to compute the norm of some wrench with both non-zero force and moment components. A little notation will be helpful.

Definition 2 *For a 2-D grasping problem, the* **grip plane** *is simply the plane of the 2-D object. For a 3-D grasping problem, the* **grip plane** *is defined as the plane containing the three contact points between the fingers and the grasped object.*

A grasping problem is characterized as 2-D if all external forces lie in the grip plane, and all external moments are perpendicular to it. If one wishes to consider other wrenches acting on a 2-D object, it should be treated as a full 3-D grasping problem.

Definition 3 *$\mathcal{W}$ denotes the entire wrench space, isomorphic to $\mathbb{R}^3$ in 2-D or $\mathbb{R}^6$ in 3-D. $\mathcal{W}_f \subset \mathcal{W}$ consists of unit forces acting in the grip plane through the center of grip, and $\mathcal{W}_{\perp f} \subset \mathcal{W}$ consists of unit forces acting perpendicular to the grip plane through the center of grip. Similarly, $\mathcal{W}_m$ and $\mathcal{W}_{\perp m}$ are the subsets of wrench space comprising pure moments in directions contained within and perpendicular to the grip plane, respectively.*

Note that $\mathcal{W}_{\perp f}$ and $\mathcal{W}_m$ only exist for the 3-D grasping problem. The center of grip is a distinguished point in the grip plane through which forces are assumed to act; precise definitions are given later. We use the term *normal finger force* to refer to the normal component of the finger force. For each finger, the total finger force is the sum of the normal finger force and the frictional force. We are now ready to define our criterion for optimum grasps.

Criterion 1 (Grasp optimality) *For a grasp g, let $Q_1(g)$ and $Q_2(g)$ measure the grasp's ability to resist unit forces in the grip plane acting through the center of grip, and unit moments normal to the grip plane, respectively. Formally,*

$$\begin{aligned} Q_1(g) &= \min_{\vec{w} \in \mathcal{W}_f} \left(\max_{\vec{f} \in g^{-1}(-\vec{w})} \frac{1}{\|\vec{f}\|_f} \right), \\ Q_2(g) &= \min_{\vec{w} \in \mathcal{W}_{\perp m}} \left(\max_{\vec{f} \in g^{-1}(-\vec{w})} \frac{1}{\|\vec{f}\|_f} \right). \end{aligned} \quad (2)$$

The optimum grasp is the one which lexicographically maximizes $(Q_1(g), Q_2(g))$, that is the grasp that maximizes $Q_2(g)$ among all grasps which maximize $Q_1(g)$.

Roughly speaking, we seek grasps which best resist forces in the grip plane. Among these, we consider the one which best resists moments perpendicular to the grip plane to be optimum. All possible external wrenches in a 2-D grasping problem can be decomposed into wrenches from $\mathcal{W}_f$ and $\mathcal{W}_{\perp m}$. This is not true for a 3-D problem; we consider resisting wrenches in $\mathcal{W}_{\perp f}$ and $\mathcal{W}_m$ in section 5.

2.1 Rounded fingertips and smoothness

The strategies for optimum grasps discussed in the following sections assume that the object to be grasped is smooth, or that the fingertips are rounded. What it important is that the direction or set of directions of the forces the finger may apply varies continuously as the contact point moves along the boundary of the object. While many common objects such as polygons or polyhedra are not inherently smooth, since a rounded fingertip can exert force in a range of directions at the vertices—independent of the coefficient of friction—the surface normal (generalized appropriately at vertices and edges) varies continuously over the boundary (see figure 1). See also [PS92], where the same assumption is made on applicable force directions at polygon vertices. Given a polyhedron P,

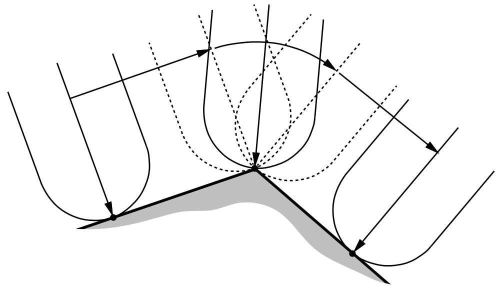

Figure 1: *With rounded fingertips, with or without friction, the normal finger force varies smoothly along the boundary of an object, even at vertices and edges.*

imagine creating a 3-D solid P' by rounding the edges and vertices of P (with an infinitesimal radius of curvature) to create an object with a well defined and continuous surface normal everywhere. Grasping P with rounded fingers is equivalent to grasping P' with "point" fingers. Since practically all real robotic fingers are rounded, our assumption is a very mild one; of course if the object to be grasped is already smooth, the rounded finger assumption is not even needed. A standard example against three finger, 3-D grasping is the tetrahedron. It may not be obvious how to achieve a three finger force closure grasp of this object (with friction), until one realizes that with rounded fingers, it is possible to grasp the object on its edges and vertices, as shown in figure 2.

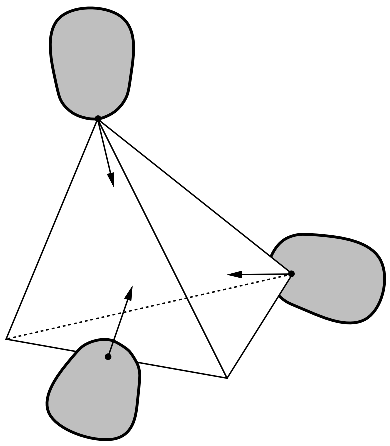

Figure 2: *Grasping a tetrahedron with three fingers.*

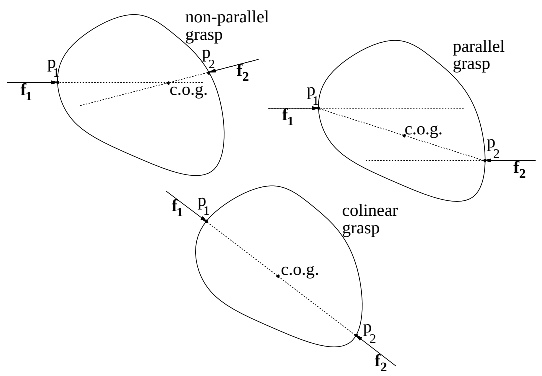

Figure 3: *The three categories of two finger, 2-D grasps, and the corresponding center of grips.*

3 Optimum two finger, 2-D grasps

The two finger, 2-D grasping problem provides a nice application of the decoupled wrench optimality criterion. The object to be grasped is a smooth planar figure, possibly non-convex. A grasp is defined by specifying two points on the perimeter where the fingers contact the object; the finger forces are directed along the inward normals at these points. The grasps fall into three categories: non-parallel, parallel, and colinear, based on the lines of action of the two normal finger forces. In non-parallel grasps, the lines of action are not parallel, in parallel grasps they are parallel but distinct, and in colinear grasps the normal finger forces share the same line of action. How the two finger center of grip is defined depends on the category of the grasp.

Definition 4 *Consider a two finger grasp in the plane, where the finger forces are applied at points p_1 and p_2 on the object, along lines of action l_1 and l_2, respectively. The* **two finger center of grip** *is defined as the intersection of l_1 and l_2 for non-parallel grasps, and as the midpoint of line segment $\overline{p_1 p_2}$ for parallel and colinear grasps. Figure 3 illustrates the various cases.*

Let us first determine what grasps maximize Q_1, that is which grasps best resist pure forces acting through the center of grip. Assume an L_1 norm on the normal finger forces for this optimization. Let $\hat{f}_1$ and $\hat{f}_2$ be unit vectors in the directions of the normal finger forces. Without friction, the set of all forces which can be applied to the object while $||\vec{f}||_f \leq 1$ is the convex hull of the origin, $\hat{f}_1$, and $\hat{f}_2$, where the $\hat{f}_i$ are treated as points by rooting the vectors at the origin (see figure 4). To account for friction, each $\hat{f}_i$ is replaced with two vectors, $\vec{f}_{il}$ and $\vec{f}_{ir}$, making an angle of ϕ with $\hat{f}_i$ and having unit projection on the latter. The set of all forces which can be applied to the object (including the frictional forces) while $||\vec{f}||_f \leq 1$ is the convex hull of the origin, plus these four vectors, as shown in figure 4. Call this region F.

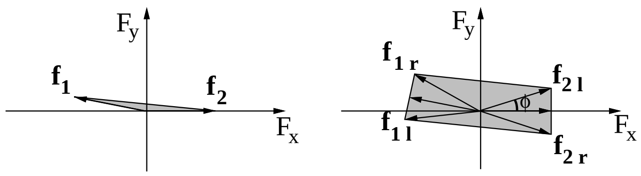

Figure 4: *The set of forces which can be applied by a two finger grasp g such that $||\vec{f}||_f \leq 1$. Left: without friction. Right: with friction, $\phi = \tan^{-1}\mu$.*

Let λF be the scaled version of region F obtained by scaling each of its defining vectors by λ. From the definition of Q_1, it follows that $Q_1(g) = \frac{1}{\lambda_0}$, where λ_0 is the minimum positive number such that $\lambda_0 F$ includes the unit circle centered at the origin. Another way of phrasing this is that $Q_1(g) = r$, where r is the radius of the largest inscribed circle in F (centered at the origin). It is obvious by inspection that to maximize $Q_1(g)$, $\hat{f}_1$ and $\hat{f}_2$ should be directly opposed. Hence grasps in the non-parallel category can never be optimum. By criterion 1, the next step is to examine the grasps in the parallel and colinear categories, and choose the one which best resists moments perpendicular to the object as the optimum grasp. The next theorem defines exactly which grasp is best.

Theorem 1 (optimum two-finger, 2-D grasp) *Let O be a planar object with smooth boundary B, and choose points p_1 and p_2 on B such that the length of chord $\overline{p_1 p_2}$ is maximized. A grasp which places the two fingers at points p_1 and p_2 is optimum under criterion 1.*

To prove theorem 1, we will first need a lemma.

Lemma 1 *Let $\overline{p_1 p_2}$ be a maximum length chord of a smooth planar object. Then the inward normals at p_1 and p_2 both lie along the chord $\overline{p_1 p_2}$.*

Sketch of proof: If the inward normal at p_i does not lie along the chord $\overline{p_1 p_2}$, then it is always possible to obtain a longer chord by sliding p_i along the perimeter of O in the proper direction. □

Proof of theorem 1: Denote the maximum chord grip by g_{mc}, and recall that $Q_2(\cdot)$ measures a grasp's ability to resist moments normal to the grip plane. We must show that $Q_2(g) \leq Q_2(g_{mc})$ for all parallel and collinear grasps g. Let μ be the coefficient of friction, and d be the distance between the contact points of a grasp.

First consider collinear grasps. The normal finger forces generate no moment, and so any externally applied moment must be completely balanced by the frictional contact forces. These forces generate a maximum couple of magnitude $\mu f d$, where f is the magnitude of the normal finger forces. To resist a unit moment, $f = \frac{1}{\mu d}$, and since $\|\vec{f}\|_f = 2f$ (assuming an L_1 norm), we obtain $Q_2(g) = \frac{1}{2}\mu d$ for any colinear grasp. Now for a parallel grasp, the maximum couple generated by the frictional forces is reduced to $\mu f d \sin\theta$, where θ is the angle between the lines of action of the frictional forces and the chord connecting the contact point. Furthermore, the normal finger forces also generate a couple, and an adversary will apply the external moment to be resisted in the same direction as this induced couple. The result is $Q_2(g) \leq \frac{1}{2}\mu d$ for any parallel grasp. By lemma 1, g_{mc} is a colinear grasp, and so $Q_2(g_{mc}) = \frac{1}{2}\mu d_{max}$, where d_{max} is the length of the maximum chord on the object. This is at least as great as the Q_2 measure for any colinear or parallel grasp, and so so g_{mc} is optimum. □

3.1 Algorithm for two finger polygon grasps

We now examine how one may compute the optimum two finger grasp of an n-sided polygon. Although the boundary of a polygon is not smooth, theorem 1 is still valid, assuming rounded fingertips (the situation is equivalent to grasping a "polygon" with "infinitesimally rounded" vertices with non-rounded fingers). By the theorem, the optimum grasp is the one which places the fingers at the endpoints of a maximum chord on the polygon. The diameter function $d(\theta)$ for a 2-D shape is simply the length of the shadow cast on (say) the x-axis when the shape is oriented at angle θ. It is easy to show that for an n-sided convex polygon, $d(\theta)$ comprises $O(n)$ piecewise sinusoids, and is computable in $O(n)$ time. Thus, the orientation which maximizes $d(\theta)$ can be found in $O(n)$ time, from which it is a simple matter to compute the maximum chord in linear time. If the polygon is non-convex the same algorithm can be used after first computing the convex hull as a pre-processing step. The maximum chord for the original polygon is the same as the maximum chord on the convex hull, since the maximum chord lies between two vertices of the hull, and all hull vertices are also vertices on the original polygon. This procedure yields an $O(n \log n)$ algorithm for finding optimum two finger grasps of non-convex polygons.

4 Optimum three finger, 2-D grasps

Next we consider optimum 2-D three finger grasps. To characterize the grasps which best resist forces acting in the plane of the object, through the center of grip, let $\hat{f}_1$, $\hat{f}_2$, and $\hat{f}_3$ be unit vectors in the directions of the three normal finger forces. Rooting these vectors at the origin and taking the convex hull of the the three endpoints and the origin gives the set of forces which can be imparted on the object, without friction. Friction is accounted for by replacing each $\hat{f}_i$ with $\vec{f}_{il}$ and $\vec{f}_{ir}$, and taking convex hull of these six vectors plus the origin. Figure 5 shows both cases. As

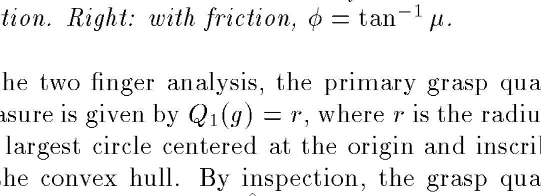

Figure 5: *The set of forces which can be applied by a three finger grasp g such that $\|\vec{f}\|_f \leq 1$. Left: without friction. Right: with friction, $\phi = \tan^{-1}\mu$.*

in the two finger analysis, the primary grasp quality measure is given by $Q_1(g) = r$, where r is the radius of the largest circle centered at the origin and inscribed in the convex hull. By inspection, the grasp quality is maximized when the $\hat{f}_i$ are positioned in symmetric fashion, with angle 120° between any two of them. Since the $\hat{f}_i$ in this case form the vertices of an equilateral triangle, we call this class of grasps *equilateral grasps*.

The optimum three finger grasp of a 2-D object must be an equilateral grasp, but there might be many such grasps. To select the optimum, we resort to the secondary quality measure: the ability to resist moments perpendicular to the plane of the forces. Consider an arbitrary equilateral grasp of a planar object, as shown in figure 6.

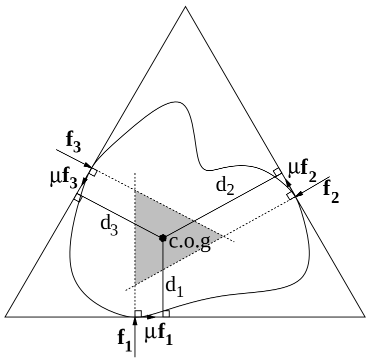

Figure 6: *The outer and inner triangles, and the center of grip for an arbitrary equilateral grasp.*

Definition 5 *For a 2-D object O and an equilateral grasp g, the* **outer triangle** *is the triangle formed by the three lines perpendicular to the normal finger forces and passing through the respective contact points. The* **inner triangle** *is the equilateral triangle formed by the intersection of the lines of action of the three normal finger forces. The* **three finger center of grip** *is the centroid of the inner triangle. If all three lines of action are coincident (degenerate inner triangle), the center of grip is their intersection point.*

Call the lengths of the sides of the outer and inner equilateral triangles s_o and s_i, respectively. If the lines of action of the normal finger forces are not coincident, so that the inner triangle is non-degenerate, it is easy to show that the forces $\vec{f_1}$, $\vec{f_2}$, and $\vec{f_3}$, must either all exert a positive moment or all exert a negative moment about the c.o.g. (this follows from the fact that the directions of the forces are 120° apart). Without loss of generality, assume the normal finger forces exert a negative (clockwise) moment about the c.o.g. as shown in figure 6. Then an adversary free to choose the external moment would apply a negative unit moment. The frictional forces must then balance both the externally applied negative moment and the negative moment induced by the normal finger forces. Assume an L_∞ norm on the finger forces. Then to maximize the largest possible moment of the frictional forces, all normal finger forces should be set to their maximum value, say f. The total moment exerted by the normal finger forces is $-3 \cdot f \cdot \frac{1}{3}(\frac{\sqrt{3}}{2}s_i) = -\frac{\sqrt{3}}{2}fs_i$. The largest possible moment exerted by the frictional forces is

$$\sum_{i=1}^{3} \mu f_i d_i = \mu f(d_1 + d_2 + d_3). \tag{3}$$

This can be simplified with a geometric result:

Lemma 2 *If p is a point in the interior of an equilateral triangle, then the sum of the distances from p to the three sides of the triangle is $\frac{\sqrt{3}}{2}s$, where s is the length of a side of the triangle.*

Sketch of proof: Straightforward geometry. A key observation is that if p moves one unit directly toward one side of the triangle, it moves exactly $\frac{1}{2}$ unit away from each of the other two sides, so that the sum of the three distances remains constant. □

By the lemma, the expression for the maximum moment induced by the frictional forces given by equation 3 reduces to $\frac{\sqrt{3}}{2}\mu f s_o$. Now letting m be the magnitude of the negative moment applied by an adversary, we have the following moment balance:

$$-\frac{\sqrt{3}}{2}fs_i + \frac{\sqrt{3}}{2}\mu f s_o - m = 0.$$

The magnitude of the largest moment which can be resisted is thus

$$m = \frac{\sqrt{3}}{2}f(\mu s_o - s_i). \tag{4}$$

Equation 4 gives a condition that an equilateral grasp must satisfy in order to resist any moments normal to the plane of the object. The ratio of side lengths between the inner and outer triangle must not exceed the coefficient of friction, otherwise the moment induced by the normal finger forces cannot be balanced by frictional forces. Moreover, equation 4 indicates how to choose an equilateral grasp with the best moment-resisting capabilities: make s_o as large as possible and s_i as small as possible. Fortunately, the next theorem shows that these goals do not conflict. In fact, in maximizing s_o, s_i is reduced to 0, its minimum value.

Theorem 2 *Let O be a 2-D object and g the equilateral grasp which maximizes the size of the outer triangle over all equilateral grasps of O. Then g is the optimum grasp of O, under criterion 1.*

Proof: Since g maximizes s_0, it suffices to show that g also minimizes s_i; equation 4 then implies optimality. We will show that all three lines of action of the normal finger forces are coincident (so $s_i = 0$) for the maximum equilateral grasp g. To consider all of the equilateral grasps on an object O, imagine O sitting in a fixed $60°$ wedge shaped cavity, as shown in figure 7. The object O contacts the two sides of the cavity at

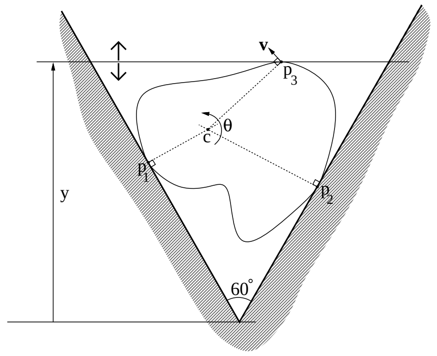

Figure 7: *The normal finger force lines of action must be coincident for the maximum equilateral grasp.*

points p_1 and p_2, and contacts a third horizontal side which is free to move up and down at the point p_3. The points p_i correspond to the points of application for an equilateral grip; as we rotate O through a complete rotation, the p_i travel over the grasp points of all possible equilateral grips. (It is not hard to show that the contact points for the maximum equilateral grasp must lie on portions of the boundary which are also on the convex hull of O.) The maximum equilateral triangle occurs when y reaches a maximum, which implies $\frac{dy}{d\theta} = 0$. Since the instantaneous center of rotation c lies at the intersection of the normals at p_1 and p_2, $\frac{dy}{d\theta} = 0$ implies p_3 lies directly above c, at which point the three normals at the p_i are coincident. These are precisely the lines of action of the normal forces of the equilateral grasp, and so $s_i = 0$. □

4.1 Algorithm for three finger polygon grasps

The optimum two finger grasp of a planar object O was computed by first finding the orientation which maximized the diameter function for O. For the three finger case, we must find the orientation θ_{max} which maximizes the O's *triameter* function $t(\theta)$, where $t(\theta)$ is proportional to the size of the unique equilateral circumscribing triangle with (say) one side parallel to the x-axis. If O is a convex polygon, then the entire orientation range $[0, 2\pi)$ may be partitioned into $O(n)$ subintervals, such that over any subinterval $t(\theta)$ is of the form $\sum_{i=1}^{3} c_i \cos(\theta + \alpha_i)$ (c_i and α_i constant). Hence θ_{max}, the orientation which maximizes the triameter function, can be found in $O(n)$ time. It is then straightforward to find the three grasp points which define the circumscribing triangle; the result is a linear time algorithm for computing the optimum three finger convex polygon grasp. As in the two finger algorithm, non-convex polygons are handled by first computing their convex hull, resulting in an $O(n \log n)$ algorithm for finding the optimum grasp.

4.2 The choice of finger force norm

We have used an L_1 norm on the finger forces to maximize the force resistance measure Q_1. Under the L_1 norm, a grasp's ability to resist forces in the grip plane is proportional to the radius of the largest inscribed circle centered at the origin and lying within the convex hull of the unit friction cones (see figures 4 and 5). As a result, parallel and colinear grasps maximize Q_1 among two finger grasps, and equilateral grasps maximize Q_1 among three finger grasps. If the L_∞ norm is used instead, these conditions are sufficient but no longer necessary for maximizing Q_1 (the difference is that Q_1 is now proportional to the radius of the largest inscribed circle lying within the Minkowski sum of the friction cones, rather than their convex hull). However, for small enough friction angles, the parallel/colinear and equilateral grasp conditions again become necessary. Proof of the following theorem may be found in [MC93].

Theorem 3 *If $\phi < 30°$, then under the L_∞ norm, parallel and colinear grasps maximize Q_1 among all two finger grasps. If $\phi < 10°$, then under the L_∞ norm, equilateral grasps maximize Q_1 among all three finger grasps.*

For example, in a three finger grasp with $\phi = 11°$, there is actually a small neighborhood of the equilateral configuration for which the Q_1 quality measure is optimized. All grasps such that three angles between the normal finger forces never exceed $123°$ resist forces in the grip plane equally well.

5 Optimum 3-D grasps

The optimum three finger grasp (under criterion 1) of a a 3-D object is a natural extension of the three finger, 2-D case. In the 3-D case, the grip plane is defined by the locations of the three finger contacts on the object, and Q_1 measures the grasp's ability to resist forces in this plane. Since the components of the total finger forces in the direction normal to the grip plane do nothing to resist forces in the grip plane or moments normal to the grip plane, the 3-D problem collapses to the 2-D problem in the grip plane.

From the analysis of the 2-D case, the best grasp is the maximum equilateral triangle grasp. In the 3-D case, to maximize Q_1, the measure of grip plane force resistance, we want to find three points on the object, such that the projections of the inward normals at these points onto the grip plane are spaced 120° apart. To maximize Q_2, the measure of resistance to moments normal to the grip plane, we want to maximize the size of the outer triangle (see figure 6), defined by the projections of the normal finger forces in the grip plane. One fact complicates the 3-D case: since the normal finger forces are not known a priori to lie in the grip plane, the magnitudes of the projections in the grip plane could be quite small, resulting in a poor grasp even when the above criteria are met. Fortunately, this is not the case; when the size of the outer triangle in the grip plane is maximized, the normal finger forces must lie within the grip plane.

Definition 6 *For a 3-D smooth object O, an* **equilateral triangular circumscribing prism** *is a prism with an equilateral triangular cross section and infinite length in its axial direction, which just encloses O. In the general case, each of the three sides of the prism will touch O at one point. For brevity, we hereafter abbreviate the term equilateral triangular circumscribing prism to simply* **circumscribing prism***.*

We will use circumscribing prisms to locate the three grasp points for an optimum three finger grasp of a 3-D object. For a given object O, there is a three dimensional collection of circumscribing prisms. To generate this set of prisms, we can consider the axis of the prism to always be parallel to the z-axis, for example, and let the orientation of O vary over $SO(3)$ (see figure 8). Since a circumscribing prism is considered

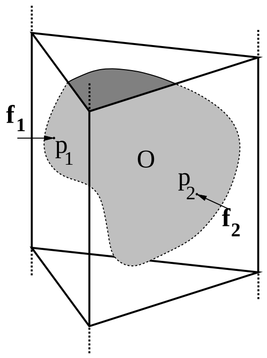

Figure 8: *A circumscribing prism for object O (point p_3 and force $\vec{f_3}$) are hidden on the back face.*

infinite in the axial direction, the maximum circumscribing prism is the one with largest cross section.

Lemma 3 *Let O be a 3-D object and P be the maximum circumscribing prism. Let p_i be the contact point on the ith face ($i = 1, 2, 3$). Then the plane defined by the p_i is normal to the axis of the prism, and the inward normals on O at the p_i all lie in that plane, and are in fact coincident.*

Sketch of proof: (See [MC93] for a more complete proof.) The proof is the 3-D analog of the proof of theorem 2. Instead of a planar object sitting in a wedge-shaped cavity with a movable third side of a triangle, we have a solid object O sitting within a circumscribing prism, having two fixed sides and one movable one. If the plane π determined by the p_i is not normal to the prism axis, it is always possible to find a rotation of O which will enlarge the size of the prism, hence π must be perpendicular to the prism axis. The inward normals at the p_i all lie within π, and must be coincident by theorem 2 (if they were not coincident, we could rotate O about the prism axis to obtain a larger circumscribing prism). □

Since the normal finger forces lie within the grip plane and are spaced 120° apart, this grasp resists forces in the grip plane better than any grasp in which the normal finger forces do not lie within the grip plane. [Except for very large coefficients of friction ($\mu > \sqrt{3}$), the convex hull of the projections of the unit friction cones into the grip plane has maximum inscribed circle when the normal finger forces lie within the grip plane.] Among grasps in which the normal finger forces lie within the grip plane and are in equilateral configuration, the maximum circumscribing prism grasp has the largest outer triangle, and therefore maximizes the Q_2 quality measure. As a result, we have:

Theorem 4 *The maximum circumscribing prism grasp is the optimum grasp for a smooth 3-D object, under criterion 1.*

5.1 Resisting moments in the grip plane

Our optimality criterion is based primarily on a grasp's ability to resist forces in the grip plane, and secondly on its ability to resist moments normal to this plane. All wrenches in 2-D case can be decomposed into wrenches from these categories. However, in 3-D there are two types of wrenches which aren't included in criterion 1: forces normal to the grip plane (wrenches in $\mathcal{W}_{f\perp}$), and moments within the grip plane (wrenches in $\mathcal{W}_m$).

The case of resisting a wrench in $\mathcal{W}_{f\perp}$ is not that interesting. Assuming the normal finger forces all lie within the grip plane, the frictional forces must provide all resistance to the external wrench. All such grasps resist forces in the direction normal to the grip plane equally well.

Resisting moments in the grip plane is more interesting. Let T be the triangle formed by the contact points of a three finger grasp g, and assume the normal finger forces lie in the grip plane. It can be shown that the hardest unit moment in the grip plane for g to resist is one which is parallel to the longest side of T, and that g's ability to resist such a moment is proportional to the length of the altitude to this side of T (see [MC93] for details). In short, g's ability to resist (worst case) moments in the grip plane is proportional to the minimum altitude of the triangle formed by the contact points. By a perturbation argument, if the contact points are not the vertices of an equilateral triangle, one of the points can always be moved slightly such that the minimum altitude of the triangle increases. It follows that the maximum *inscribed* equilateral triangle defines the optimum grasp for this case. Figure 9 illustrates two optimum grasps of an ellipse in the plane. The maximum circumscribing equilateral triangle grasp is based on our familiar criterion 1; the maximum inscribed equilateral triangle grasp is optimum for resisting moments within the plane of the ellipse. The latter generalizes naturally to a 3-D grasp based on the maximum inscribed equilateral prism.

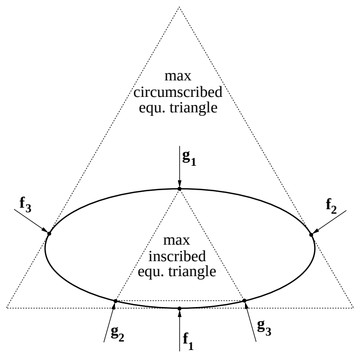

Figure 9: *Two optimum grasps for an ellipse, under different criteria. The $\vec{f}_i$ grasp is good at resisting moments normal to the plane of the ellipse; the $\vec{g}_i$ grasp is good at resisting moments within the plane.*

5.2 Algorithm for three finger polyhedron grasps

Consider the special problem of optimally grasping polyhedra. By a perturbation argument, the optimum grasp points must be vertices of the polyhedron (if an edge or face of the polyhedron lie on one side of a circumscribing prism, it is always possible to rotate the polyhedron slightly to obtain a larger circumscribing prism). Furthermore, by lemma 3 the maximum circumscribing prism must have an axis perpendicular to the plane of the grasp points, and so given any three vertices of the polyhedron to be grasped, the size of the corresponding circumscribing prism can be checked in constant time. Naively testing all triples of vertices of the polyhedron gives an $O(n^3)$ algorithm for finding the optimum three finger grasp of a polyhedron; a tighter bound is likely possible.

6 Conclusion

Using a quality measure based on decoupled wrenches, we have found optimum grasps of 2-D objects (two or three fingers) and 3-D objects (three fingers). Simple and efficient algorithms have been presented for grasping convex and non-convex polygons, as well as polyhedra. The same theory applies to other objects as well, and optimal grasps can be computed for many other shapes, such as ellipsoids. In addition, many complicated objects can be approximated by polygons or polyhedra. Qualitatively, these grasps seem to be the "right" ones; they resemble the way a human might grasp an object. Furthermore, they are useful in practice, where typical applications involve a small number of rounded fingertips, with friction present at the contact points.

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