CS184 Review Quiz Solutions

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1. Let $u_i = [0, \ldots, 0, 1, 0, \ldots, 0]$, where 1 is at the i-th position. Let v_j be defined similarly. Then, we can see that $u_i^T A v_j = a_{ij}$ and $v_j^T A u_i = a_{ji}$. Since we are given $u_i^T A v_j = v_j^T A u_i$, $a_{ij} = a_{ji}$ for all $1 \le i \le n$ and $1 \le j \le n$. Therefore, $A = A^T$.

2. By the definition of cross product,

$$b \times c = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where i, j, k are the unit vectors, $i = (1,0,0)$, $j = (0,1,0)$ and $k = (0,0,1)$.

$$a \cdot b \times c = (a_1 i + a_2 j + a_3 k) \cdot \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

3. (a) Since cross product has the distributive property,

$$f(v + w) = u \times (v + w) = u \times v + u \times w = f(v) + f(w)$$

You can also "factor out" a scalar value from a cross product,

$$f(\lambda v) = u \times \lambda v = \lambda(u \times v) = \lambda f(v)$$

Therefore, $f(v) = u \times v$ is a linear function of v.

(b) $u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)i + (u_3 v_1 - u_1 v_3)j + (u_1 v_2 - u_2 v_1)k$.

In matrix form, this is equal to $\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = Av.$

4. We assume that A is a real matrix. Let v be an eigenvector of $A^T A$. Let λ be its corresponding engenvalue. Then,

$$A^T Av = \lambda v$$

Left-multiply each side by $\overline{v}^T$, where $\overline{v}^T$ is the conjugate transpose of v. We now have

$$\overline{v}^T A^T Av = \overline{v}^T \lambda v$$

Notice that $A^T = \overline{A}^T$ since A is a real matrix. Therefore,

$$\overline{v}^T \overline{A}^T Av = \lambda \overline{v}^T v$$

$$\overline{(Av)}^T Av = \lambda \overline{v}^T v$$

Notice that $\overline{(Av)}^T Av = ||Av||^2$, which is non-negative. Similarly, $v^T v = ||v||^2$, which is also non-negative. Since

$$||Av||^2 \ = \lambda ||v||^2$$

λ is non-negative. Therefore, all eigenvalues of $A^T A$ are non-negative.