Solution 1. Consider a final stable marriage. Because all the males have the same preference ordering, then we can assign each female a unique number \( k \) representing her ranking on the lists. Female \( k \) has a spouse, and she must be the \( k \)-th on his ordering list. This male must have proposed \( k \) times. So in total, all males together proposed \( \sum_{1 \leq k \leq n} k = n(n + 1)/2 \) times.

Solution 2. From the definition of expected values, we obtain \( E[X] = \sum_{1 \leq k \leq n} \Pr[X = k] \cdot k \).

Because \( \Pr[X \geq k] = \sum_{k \leq j \leq n} \Pr[X = j] \), we have

\[
\sum_{1 \leq k \leq n} \Pr[X \geq k] = \sum_{1 \leq k \leq n} \sum_{k \leq j \leq n} \Pr[X = j] \\
= \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq j} \Pr[X = j] \\
= \sum_{1 \leq j \leq n} \Pr[X = j] \cdot j
\]

Hence, \( E[X] = \sum_{1 \leq k \leq n} \Pr[X \geq k] \).

Solution 3. There are \( n! \) possible true permutations, and \( n^n \) possible \( n \)-permutations. Each permutation is equally likely. So the probability that a random \( n \)-permutation is a true permutation is \( \frac{n!}{n^n} \sim \frac{\sqrt{2\pi n}}{e^n} \).

Solution 4. (a) Think of this as labelling every ball from 1, \ldots, \( m \) with a bin number from \( \{1, \ldots, n\} \). There are \( n^m \) such labellings, and each is equally likely (probability \( (1/n)^m \)). The number of labellings which have \( m_1 \) balls in bin 1 is \( \binom{m}{m_1} \) which can be specified by the subset of balls which receive a bin number “1”. So the probability is

\[
\binom{m}{m_1} \left( \frac{1}{n} \right)^m = \frac{m!}{m_1! m_2!} \left( \frac{1}{n} \right)^m
\]

(b) If we apply the same argument twice, the number of labellings with \( m_1 \) ones is \( \binom{m}{m_1} \) and there are \( m - m_1 = m_2 + m_3 \) balls labelled “not 1”. We can further divide the balls labelled “not 1” into
groups labelled two and three. The total probability of the specified counts in bins 1, 2, 3 is

\[ \binom{m}{m_1} \binom{m - m_1}{m_2} \left( \frac{1}{n} \right)^m = \frac{m!}{m_1! \cdot m_2! \cdot m_3!} \left( \frac{1}{n} \right)^m \]

(c) Continuing inductively, The probability that the distribution of \( m \) balls into \( n \) bins giving \( m_1, \ldots, m_n \) balls in bin 1 through bin \( n \), where \( m_1 + \cdots + m_n = m \), is

\[ \binom{m}{m_1} \binom{m - m_1}{m_2} \cdots \binom{m - m_1 - \cdots - m_{n-2}}{m_{n-1}} \left( \frac{1}{n} \right)^m = \frac{m!}{m_1! \cdots m_n!} \left( \frac{1}{n} \right)^m \]

Notice that this result contains parts (a) and (b) as special cases. That is, if we substitute \( m_n = m_{n-1} = \cdots m_4 = 0 \) into this formula, since \( 0! = 1 \), it simplifies to either (a) if \( m_3 = 0 \) or (b) if \( m_3 \neq 0 \).