1. (a) This is 10 Bernoulli trials. So the expected number of heads is $0.8 \times 10 = 8$.

(b) In a random permutation, 1, 2, 3 can have $3! = 6$ total number of possible configurations. There are only two possible configurations if we require 1 and 2 both come before 3 in the permutation. Each configuration is equally likely. So the probability in the question is $2/6 = 1/3$.

(c) Because $X$ and $Y$ are two independent random variables, $E[XY] = E[X] \cdot E[Y] = 4 \times 4 = 16$.

(d) $\ln n$.

2. Because $X$ has approximately a Poisson distribution with parameter $\lambda = 1$, $E[X] = \text{Var}[X] = 1$.

(a) $\Pr[X \geq 5] \leq \frac{E[X]}{5} = \frac{1}{5}$.

(b) $\Pr[X \geq 5] \leq \Pr[|X - 1| \geq 4] \leq \frac{1}{16}$.

(c) $\Pr[X \geq 5] = \Pr[X \geq (1 + 4) \cdot 1] < e^{-4}$, because $\delta = 4 < 2e - 1$.

(d) This is a two-step argument. First of all, $X$ has approximately a Poisson distribution. This does not mean that $X$ is a sum of Poisson trials. But the binomial distribution is a sum of Bernoulli trials and is well-approximated by a Poisson distribution. In other words $X \approx Y \approx Z$ where $X$ is the number of fixed points, $Y$ is a Poisson r.v., and $Z$ is a binomial r.v. Chernoff can be applied to $Z$.

3. Details of the derivation are in the lecture notes.

(a) This is the coupon collector problem. $n \ln n$.

(b) This is the birthday-paradox problem. $\sqrt{2n}$.  

(c) Expected number of proposals a male makes = ln \( n \), expected rank of his final spouse = ln \( n \).

(d) Expected number of proposals a female receives = ln \( n \), expected rank of her final spouse = \( \frac{n}{\ln n} \).

4. (a) \( n \ln n + \Omega(n) \).

(b) Approximately, we can consider the process of generating random graphs of \( n \) vertices as throwing balls into \( n \) bins. Adding an edge is equivalent to throwing two balls into the bins. So when we add \( n \) edges, i.e., throw \( 2n \) balls, the expected number of empty bins is \( n(1 - \frac{1}{n})^{2n} = \frac{n}{e^2} \). An empty bin means a single vertex that touches no edges, and is therefore an isolated connected piece. So the expected number of connected pieces is at least \( \frac{n}{e^2} + 1 \).

(c) Let \( Y_i \) denote the number of edges we need to add to make the graph change from \( i \) connected pieces to \( i - 1 \) connected pieces. From the lecture notes, the probability of adding an edge that can change the graph from \( i \) connected pieces to \( i - 1 \) connected pieces is greater than or equal to \( (i - 1)/(n - 1) \).

Let \( X_i \) denote the number of balls we need to throw into \( n - 1 \) bins to change the number of empty bins from \( i - 1 \) to \( i - 2 \). So the probability of adding a ball that can change the number of empty bins from \( i - 1 \) to \( i - 2 \) is \( (i - 1)/(n - 1) \). So \( \Pr[X_i \geq k] \geq \Pr[Y_i \geq k] \). Because \( E[X_i] = \sum_k \Pr[X_i \geq k] \), and \( E[Y_i] = \sum_k \Pr[Y_i \geq k] \), \( E[X_i] \geq E[Y_i] \). As we have computed in class, when we throw \( n \) balls into \( n \) bins, the expected number of empty bins is \( n/e \). Let \( T_1 \) be the number of epochs the random graph goes through when we throw in \( n \) edges, which is the number of connected pieces in the graph. Let \( T_2 \) be the number of empty bins when we throw in \( n \) balls into \( n \) bins. Because \( E[X_i] \geq E[Y_i] \), we can see that \( \Pr[T_1 \geq k] \leq \Pr[T_2 \geq k] \), which implies \( E[T_1] \leq E[T_2] = n/e \). So the upper bound is \( n/e \).