On the Reducibility of Submodular Functions

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Problem Statement

1. Submodular function

- Given a finite set $N = \{1, 2, \ldots, n\}$
- Given a set function $f : 2^N \mapsto \mathbb{R}$

$$f(i|A) \geq f(i|B),\ \forall A \subseteq B \subseteq N,\ \forall i \in N \setminus B$$

"diminishing return property"

$f(i|A) \triangleq f(A \cup \{i\}) - f(A)$ is called "marginal gain"

2. Unconstrained optimization

$\text{Problem 1}: \min_{X \subseteq N} f(X)$ polynomial and inefficient: $\mathcal{O}(n^6)$

$\text{Problem 2}: \max_{X \subseteq N} f(X)$ NP-hard

Investigate some properties that…

1. can scale up the optimization
2. do not incur performance loss

Reducibility

1. (Set interval) lattice

Given $A \subseteq B \subseteq N$, $[A, B] \triangleq \{S \mid A \subseteq S \subseteq B\}$

2. Trivial (largest) lattice $[\emptyset, N]$

Suppose $f(X_*) = \min_{X \subseteq N} f(X),\ f(X^*) = \max_{X \subseteq N} f(X)$

$$X_* \in [\emptyset, N],\ X^* \in [\emptyset, N]$$

3. Reducibility

Find a smaller lattice $[X, Y] \subset [\emptyset, N]$

$$X_* \in [X, Y],\ X^* \in [X, Y]$$

Reduction Algorithms

Algorithm 1 Reduction for Problem 1

Input: $f, [X_0, Y_0] \leftarrow [\emptyset, N], t \leftarrow 0$.
Output: $[X_t, Y_t]$.
1: $U_t = \{i \in Y_t \setminus X_t \mid f(i|X_t) < 0\}$.
2: $X_{t+1} \leftarrow X_t \cup U_t$.
3: $D_t = \{j \in Y_t \setminus X_t \mid f(j|Y_t - j) > 0\}$.
4: $Y_{t+1} \leftarrow Y_t \setminus D_t$.
5: If $[X_{t+1}, Y_{t+1}] = [X_t, Y_t]$, terminate.
6: $t \leftarrow t + 1$. Go to Step 1.

Algorithm 2 Reduction for Problem 2

Input: $f, [X_0, Y_0] \leftarrow [\emptyset, N], t \leftarrow 0$.
Output: $[X_t, Y_t]$.
1: $U_t = \{i \in Y_t \setminus X_t \mid f(i|X_t) < 0\}$.
2: $Y_{t+1} \leftarrow Y_t \setminus U_t$.
3: $D_t = \{j \in Y_t \setminus X_t \mid f(j|Y_t - j) > 0\}$.
4: $X_{t+1} \leftarrow X_t \cup D_t$.
5: If $[X_{t+1}, Y_{t+1}] = [X_t, Y_t]$, terminate.
6: $t \leftarrow t + 1$. Go to Step 1.

Proposition:

- For Algorithm 1, $X_* \in [X_t, Y_t] \subseteq [\emptyset, N],\ \forall t$
- For Algorithm 2, $X^* \in [X_t, Y_t] \subseteq [\emptyset, N],\ \forall t$

Bad news:

For many functions, Algorithm 1 & 2 output $[X_0, Y_0] = [\emptyset, N]$

Irreducible Function

A submodular function that makes Algorithm 1 & 2 vacuous

Proposition:

Given f is submodular, f is irreducible if and only if

$$f(i|\emptyset) > 0,\ f(i|N - i) < 0,\ \forall i \in N$$

Perturbation Reduction Optimization

Main idea: Adding noise to perturb the marginal gains

$\text{Problem 3}: \min_{X \subseteq N} g(X) \triangleq \min_{X \subseteq N} f(X) + r(X)$

$\text{Problem 4}: \max_{X \subseteq N} g(X) \triangleq \max_{X \subseteq N} f(X) + r(X)$

r is modular function, $r(X) \triangleq \sum_{i \in X} r(i)$, so g is submodular

Algorithm:

1. $[X_0, Y_0] \leftarrow [\emptyset, N]$. Run Algorithm 1/2 for f, $[S, T] \leftarrow [X_t, Y_t]$
2. (Perturbation) Given t, generate r. Let $g = f + r$
3. (Reduction) $[X_0, Y_0] \leftarrow [S, T]$, Run Algorithm 1/2 for g, $[S, T] \leftarrow [X_t, Y_t]$
4. (Optimization) Solve $X_*^p \in \arg\min_{X \in [S,T]} f(X)$, or $X_p^* \in \arg\max_{X \in [S,T]} f(X)$

Theoretical Analysis

Reduction rate: $1 - \frac{|Y_t \backslash X_t|}{n} \in [0, 1]$

Theorem 1: Suppose $r(i)$ is uniformly generated from $[-t, t]$ with $t > 0,\ \forall i \in N$. Then the reduction rate $R_t \geq 1 - \frac{C_f}{2tn}$

Performance loss: $f(X_*^p) - f(X_*) > 0,\ f(X^*) - f(X_p^*) > 0$

Theorem 2:

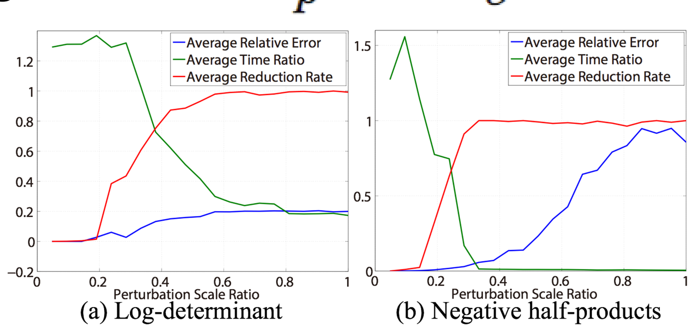
$$f(X_*^p) - f(X_*) < -r(X_t \setminus X_*) + r(X_* \setminus Y_t) < ntR_t$$
$$f(X^*) - f(X_p^*) < r(X_t \setminus X^*) - r(X^* \setminus Y_t) < ntR_t$$

Experimental Results

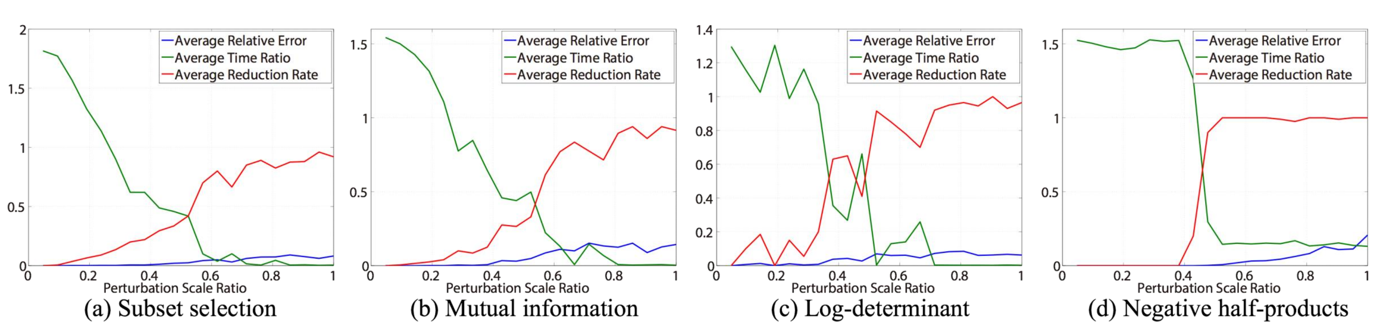
For irreducible functions, we use exact solvers in Step 4

Criteria: Denote the outputs of the proposed method and existing method X^p and X^e. Running time are T_p and T_e

- Relative error: $E_r \triangleq \frac{|f(X^e) - f(X^p)|}{|f(X^e)|}$
- Time ratio: T_p / T_e
- Reduction rate: $1 - \frac{|Y_t \backslash X_t|}{n}$



Minimization



Maximization