Control Capacity

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***Abstract*—This paper presents a notion of "control capacity" that gives a fundamental limit on the control of a system through an unreliable actuation channel. It tells us how fast we can reliably actively dissipate uncertainty in a system through that actuation channel. We give a computable single-letter characterization for scalar systems with memoryless stationary multiplicative actuation channels. The sense of control capacity is tight for answering questions of stabilizability for scalar linear systems — a system is stabilizable through an actuation channel if and only if the control capacity of that actuation channel is larger than the log of the unstable open-loop eigenvalue. For second-moment senses of stability, our result recovers the classic uncertainty-threshold principle result. However, our formulation can also deal with any other moment. The limits of higher and higher moment senses of stability correspond to a "zero-error" sense of control capacity and taking the limit to weaker-and-weaker moments corresponds to a "Shannon" sense of control capacity.**

***Index Terms*—control, capacity, anytime, zero-error, stabilization**

I. Introduction

Communication capacity is used as a standard metric in information transmission systems. Previous results have shown that information flows and bottlenecks in systems clearly affect our ability to control and stabilize a system [1], [2], [3]. These works treated the uncertainty in the plant as an information *source*. The data-rate theorems [1], [4], [5] are a product of using information-theoretic techniques to understand the impact of unreliable communication channels (connecting the sensor to the controller) on the ability to control a system. Anytime results [2] also focused on unreliable communication channels, but focused on reliability aspects of the sensing channel (i.e. the channel carrying sensor measurements about the plant state to the controller). This highlighted the differences between delay in control and communication: *when* a bit is reliably learned matters in control systems, and the degree to which it matters depends on the sense of stability desired.

In addition to the source nature of information in control systems, we find there is also a *sink* nature to systems[1]. Control systems can reduce uncertainty about the world by moving the world to a known point: it is this dissipation of information/uncertainty that we refer to as the sink nature of a control system [8]. To understand this further, this paper focuses on the actuation channels in the system (Fig. 1), where the unreliability is about the control action and not the observation.

[1]Work on Witsenhausen's counterexample shows that there is also a *channel* aspect to decentralized control systems [6], [7].

Unreliability in control systems has been extensively studied, starting with the uncertainty threshold principle [9], which considers simultaneously varying system and control gains (i.e. actuation channels). [3] focuses exclusively on erasure actuation channels with a dynamic programming approach. [10], [11] consider erasure actuation channels in context of intelligent actuators with decoders at the plant, which allows for greater flexibility in control design. More recently, [12] focuses informational bottlenecks due to varying system gain, which is also a closely related problem to the one considered here.

Information theory understands the capacity of a communication channel as the maximum number of bits of information that can be transmitted across the channel in a unit timestep. So, what is the maximum number of bits of the system state that can be dissipated by a controller in a single timestep? Is it important which bits are dissipated? It turns out the answer to the second question, in the context of control systems, is yes. To stabilize a system, the controller must dissipate the uncertainty in the most-significant bits (MSBs) of the state. The maximum number of these MSBs that can be dissipated is the "control capacity."

The work in this paper is also motivated by ideas in portfolio theory. There, the key concept is the doubling rate, i.e. the rate at which a gambler who chooses an optimal portfolio doubles his principal [13]. Control systems, like portfolios, have an underpinning of exponential growth. Just as the investor can choose to buy and sell at each time step to maximize growth, the controller has the choice of control strategy to minimize growth (or maximize decay). The "tolerable growth rate" for a control system can be thought of as a parallel to the doubling rate in portfolio theory. We find that a larger control capacity means that the system can tolerate a higher intrinsic rate of uncertainty growth. This also parallels the classic sense in communication channels, where the rate is how fast the set of potential messages grows with blocklength.

There are two big advantages to the information-theoretically motivated definitions of control capacity here. First, they more easily allow us to measure the impact of side information to improve performance in control systems. This is explored in [14] but could not be included here due to space limits. Second, they allow us to move beyond second-moment notions of state stability which dominate all previous works. We define a range of notions of capacity corresponding to different moments of stability. The weakest notion of stability is given by the logarithmic decay rate of the system, and we call the related notion of capacity "Shannon" control capacity. This is inspired by the results on the anytime capacity of a channel, where it turns out that as the desired moment sense of stability for the system goes to zero, the anytime capacity converges to the Shannon capacity of the corresponding communication channel [2]. The strictest notion is zero-error control capacity, which is similar to the worst-case perspective of robust control. The classic second-moment notion of stability lies in between these two.

Unreliable actuation channels may arise in the context of wireless control systems, where dumb actuators receive control signals over packet-dropping networks or fast-fading channels. Bio-medical technologies such as brain-machine interfaces will have to rely on wirelessly connected low-power unreliable actuators and sensors, and moreover the biological actuation system has its own eccentricities. These issues also emerge in the development of "wearable" technologies, and more broadly speaking in the development of the Internet of Things (IoT). This paper takes a look at simplified multiplicative models with i.i.d. randomness to get a theoretical understanding of the basic issues. Philosophically, these can be viewed as the counterparts of DMC channels in traditional communication, or perhaps more precisely, general additive noise channels.

II. Actuation Channels

Consider a simple scalar control system with perfect state observation, and no additive driving disturbance as in [9].

$$\begin{aligned} X[n+1] &= a(X[n] + B[n]U[n]), \\ Y[n] &= X[n]. \end{aligned} \tag{1}$$

Given a discrete-time system with state $X[n]$, the actuation channel is what combines the applied control to the current state before the state evolution takes it to the next state. For linear systems, this is captured in the block $(X[n]+B[n]U[n])$ in the dynamics. Here, the $B[n]$ represents the uncertainty in how the applied control will actually influence the state. We will model these B as iid random.

As this is a very basic model, it has been studied in the literature, but in control, it is almost exclusively studied in the mean-squared sense of stability.

Suppose $B[n]$ are a sequence of i.i.d. random variables with mean μ_B and variance σ_B^2, and $X[0] \sim \mathcal{N}(0,1)$. The system state is scaled up by a scalar constant factor a at each time step after the control is applied. The aim is to choose $U[n]$, a function of $Y[n]$, so as to stabilize the system.

The classic uncertainty threshold principle gives the achievable strategy and converse for this system for mean-squared error and shows that linear strategies are optimal in this case [9].

Example 2.1: System (1) is mean-square stabilizable using linear strategies if $a^2 < \left(1 + \frac{\mu_B^2}{\sigma_B^2}\right)$.

Let $U[n] = \frac{k}{a} \cdot Y[n] = \frac{k}{a} \cdot X[n]$, where k is a scalar constant. Then,

$$\begin{aligned} \mathbb{E}[X[n+1]^2] &= \mathbb{E}[(a + kB[n])^2 X[n]^2] \\ &= (a^2 + 2ak\mu_B + k^2(\mu_B^2 + \sigma_B^2))\mathbb{E}[X[n]^2] \end{aligned} \tag{2}$$

The scale factor, $(a^2 + 2ak\mu_B + k^2(\mu_B^2 + \sigma_B^2))$, is minimized at $k = -\frac{a\mu_B}{\mu_B^2+\sigma_B^2}$. Thus, with a linear strategy, the minimum growth rate is given by $a^2 \frac{\sigma_B^2}{\mu_B^2+\sigma_B^2}$, which is smaller than 1 if $a^2 < \left(\frac{\mu_B^2+\sigma_B^2}{\sigma_B^2}\right)$. ■

The threshold here shows that the randomness in the control parameter $B[n]$ is the "bottleneck" on the controller's ability to stabilize the system. To tolerate a larger system growth a we need to somehow reduce the randomness in $B[n]$, or change what we mean by stability.

III. Carry-free systems

To gain an intuition into the uncertainty-threshold principle, we build on the "carry-free" models introduced in [15], [16] which generalize ADT deterministic-models [17]. Unfortunately, we do not have the space to describe the models here, but the setup is described in detail in [18], [14]. Consider the carry-free system, $\mathcal{S}$, with a random gain for the control input, with unit system gain (Fig. 1).

$$x[n+1](z) = x[n](z) + b[n](z) \cdot u[n](z) \tag{3}$$
$$y[n](z) = x[n](z). \tag{4}$$

$$\begin{aligned} b[n](z) = 1 \cdot z^{g_{det}} + 0 \cdot z^{g_{det}-1} + 0 \cdot z^{g_{det}-2} + \cdots \\ + b_{g_{ran}}[n] \cdot z^{g_{ran}} + b_{g_{ran}-1}[n] \cdot z^{g_{ran}-1} + \cdots \end{aligned}$$

Basically, b is a 1, followed by 0's, followed by random numbers. g_{det} is the highest deterministic level for b and g_{ran} is the highest random level. So, there are $g_{det} - g_{ran}$ deterministic bits in b.

Let the degree of $x[n]$ be denoted as d_n. Our aim is to understand the stability of this system, which is captured by the degree d_n. First, we define the notion of "in-the-box" or zero-error stability, where the degree of the state remains bounded with probability 1.

Definition 3.1: The system eq. (3) is stablizable in the zero-error sense if there exists a control strategy $u[\cdot](z)$ such that $\exists N, M < \infty,$ s.t. $\forall n > N, d_n < M$ with probability 1.

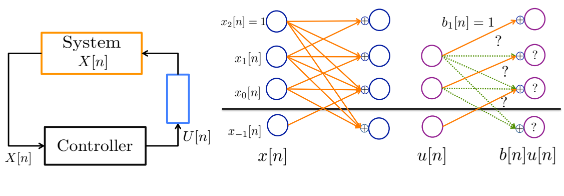

Fig. 1. On the left, we can see where the actuation channel is. On the right, we have a carry-free model of this with system gain 1 (Eq. (3)).

The picture clearly illustrates the ideas that we want to capture with control capacity. How many bits-levels can we reduce the degree by with probability 1? This should be the zero-error control capacity, which we define formally as follows.

Definition 3.2: The zero-error control capacity of the system $\mathcal{S}$ from eq. (3) is defined as the largest constant $C_{ze}(\mathcal{S})$ such that there exists a control strategy $u[0](z), \ldots, u[n](z)$ such that

$$P\left(\frac{1}{n}(d_0 - d_n) \geq C_{ze}(\mathcal{S})\right) = 1. \quad (5)$$

The time index n does not matter.

C_{ze} is essentially the largest decay exponent that is possible for the system state.

Theorem 3.1: Consider the system $\widetilde{S}^a$:

$$\widetilde{x}[n+1](z) = a(z)(\widetilde{x}[n](z) + b[n](z) \cdot \widetilde{u}[n](z)), \quad (6)$$

where $a(z)$ is a constant known polynomial with gain g_a.

Also consider the affiliated system $\mathcal{S}$ defined above in eq. (3). Then, $\exists N, M < \infty$, s.t. $\forall n > N, d_n < M$ with probability 1 if and only if $C_{ze}(\mathcal{S}) \geq g_a$.

The zero-error control capacity of the the carry-free system is determined by the number of deterministic bits in the gain $b[n](z)$.

Theorem 3.2: The zero-error control capacity for the system defined by eq. (3) is given by

$$C_{ze}(\mathcal{S}) = g_{det} - g_{ran}. \quad (7)$$

The formal proofs of these theorems are found in [14] (Thm. 5.3.6, p. 99 and Thm. 5.3.8, p.101) and are omitted because of space constraints.

IV. Real zero-error control capacity

Inspired by the carry-free models, we can define what zero-error control capacity should be for real-valued systems.

Consider the following system $\mathcal{S}(p_{B[0]}, p_{B[1]}, \ldots)$, with system dynamics $a = 1$.

$$\begin{aligned} X[n+1] &= X[n] + B[n]U[n] \\ Y[n] &= X[n] \end{aligned} \quad (8)$$

The control signal $U[n]$ can causally depend on $Y[i], 0 \leq i \leq n$. Let $\sigma(Y_0^n)$ be the sigma-algebra generated by the observations. Then $U[n]$ can be a function of these random variables, i.e. $U[n] \in \sigma(Y_0^n)$. The random variables $B[i], 0 \leq i \leq n$ are independent, and $B[i] \sim p_{B[i]}$. The distribution of each $B[i]$ is known to the controller beforehand. Consider $X[0]$ to be a random variable with density $p_{X[0]}(\cdot)$ and no atom at 0.

Definition 4.1: The zero-error control capacity of the system $\mathcal{S}(p_{B[0]}, p_{B[1]}, \cdots)$ is defined as

$$C_{ze}(\mathcal{S}) = \max\left\{C \middle| \exists N, \text{ s.t. } \forall n > N, \max_{U_1^n \text{ s.t. } U[i](\cdot) \in \sigma(Y_0^i)} P\left(-\frac{1}{n}\log\left|\frac{X[n]}{X[0]}\right| > C\right) = 1\right\}.$$

This is a well defined quantity since $p_{X[0]}(\cdot)$ has no atoms at 0. It could be infinite however.

Consider also the system $\widetilde{\mathcal{S}}^a(p_{B[0]}, p_{B[1]}, \cdots)$, with no additive disturbances and noise but with non-trivial intrinsic growth a,

$$\begin{aligned} \widetilde{X}[n+1] &= a \cdot (\widetilde{X}[n] + B[n]\widetilde{U}[n]) \\ \widetilde{Y}[n] &= \widetilde{X}[n] \end{aligned} \quad (9)$$

with initial condition $\widetilde{X}[0]$ identical to system $\mathcal{S}$ (i.e. such that $p_{\widetilde{X}[0]}(\cdot)$ has bounded support and no atom at 0). Again, $\widetilde{U}[n] \in \sigma(\widetilde{Y}_0^n)$. The random variables $B[i]$ in the system have the same distributions as those of system $\mathcal{S}$.

For this system, we define a notion of stability in a box.

Definition 4.2: The system is said to be stabilizable in the **zero-error** sense if $\exists N, M < \infty$ and a causal control strategy U_0^∞ such that $U[i](\cdot) \in \sigma(Y_0^i)$ $\forall i$, such that $\forall n > N$,

$$P\left(|\widetilde{X}[n]| < M\right) = 1. \quad (10)$$

Theorem 4.1: Consider a system with bounded initial condition, i.e. $p_{X[0]}(\cdot)$ has bounded support and $|X[0]| < \gamma$ for some $0 < \gamma < \infty, \gamma \in \mathbb{R}$.

(1) If the zero-error control capacity of the system $\mathcal{S}(p_{B[0]}, p_{B[1]}, \cdots)$ (eq. (8)),

$$C_{ze}(\mathcal{S}) > \log|a|, \quad (11)$$

then the associated system $\widetilde{\mathcal{S}}^a(p_{B[0]}, p_{B[1]}, \cdots)$, with the identical initial condition $\widetilde{X}[0] = X[0]$, and with the same distributions for the random control gains $B[i]$, is stabilizable in a zero-error sense.

(2) Further, for the system $\widetilde{\mathcal{S}}^a(p_{B[0]}, p_{B[1]}, \cdots)$, if $\exists 0 < N, M < \infty$ and a causal control strategy U_0^∞ such that $U[i](\cdot) \in \sigma(Y_0^i)$ such that $\forall n > M$, $P(|\widetilde{X}(n)| < M) = 1$, then

$$\forall \epsilon > 0, C_{ze}(\mathcal{S}) \geq \log|a| - \epsilon, \quad (12)$$

The fact that this theorem is true, namely the fact that the zero-error control capacities for system $\mathcal{S}(p_{B[0]}, p_{B[1]}, \cdots)$ lets us answer a question about the system $\widetilde{\mathcal{S}}^a(p_{B[0]}, p_{B[1]}, \cdots)$, justifies thinking of the control capacity as a property of the actuation channel.

The proof is omitted for space constraints, but is available in [14].

A. Calculating zero-error control capacity

The operational definitions for zero-error control capacity involve an optimization over an infinite sequence of potential control laws that are allowed to depend on the entire past. Here we show that in fact this quantity can be easily computed in the case where the $B[n]$'s are i.i.d. over time. If $p_B(\cdot)$ does not have bounded support, the zero-error control capacity is 0.

Theorem 4.2: The zero-error control capacity of the system eq. (8), $\mathcal{S}(p_B)$ (with i.i.d. $B[n]$'s and hence parameterized by the single distribution $p_B(\cdot)$), where $p_B(\cdot)$ does not have bounded support is zero.

Proof: Consider any strategy $U_0^{n-1}(\cdot)$. Then, for every value of $X[n-1]$ and chosen $U[n-1] \neq 0$ there exists a realization of $B[n]$ that just pushes it over any boundary, i.e., $\forall\ M < \infty$, $P(|X[n]| > M) > 0$. If all $U[i] = 0$ for $0 \leq i \leq n-1$ then the capacity is zero trivially, since $X[0] = X[n]$. ■

Theorem 4.3: Consider the system $\mathcal{S}(p_B)$ (parameterized with a single distribution $p_B(\cdot)$ since the $B[n]$'s are i.i.d.)

given by eq. (8), with i.i.d $B[n] \sim p_B(\cdot)$, and where $p_B(\cdot)$ is defined with essential support $[b_1, b_2]$. b_1 is the essential infimum and b_2 is the essential supremum of the distribution of the random variables $B[n]$. Then the zero-error control capacity of the system is

$$C_{ze}(\mathcal{S}) = \begin{cases} \log\left|\frac{b_2-b_1}{b_2+b_1}\right| & \text{if } \ 0 \notin [b_1, b_2] \\ 0 & \text{if } \ 0 \in [b_1, b_2] \end{cases} \tag{13}$$

Remark 4.1: This proof (found in [14]) reveals that there is nothing lost by going to linear strategies for the zero-error control capacity problem. The core reason is a key lemma which lets us decouple the impact of the control at each time step. Furthermore, the fact that linear strategies suffice tells us that the zero-error control capacity result immediately generalizes to the case with bounded additive disturbances W and bounded additive noise V. The converse clearly holds because the additive terms can be given by a genie. The achievability also holds because the system in closed-loop becomes linear time-varying (where the time-variation is random). Because the zero-error capacity is strictly greater than $\log(a)$, the closed-loop dynamics will take all the initial conditions to zero exponentially fast. Where there are additive disturbances and noises, the resulting system state is a superposition of the responses to bounded inputs. The system is thus BIBO stable.

V. "Shannon" control capacity

Here we consider the stability of the logarithm of the state.

Definition 5.1: The system (9) is said to be **logarithmically stabilizable** if there exists a causal control strategy $U_0^n(\cdot) \in \sigma(Y_0^n)$ such that $\exists M \in \mathbb{R}$, $M < \infty$, s.t. $\mathbb{E}[\log|\widetilde{X}[n]|] < M$ for all n.

Definition 5.2: The control capacity in the Shannon-sense of the system $\mathcal{S}(p_{B[0]}, p_{B[1]}, \cdots)$ in eq. (8) is defined as

$$C_{sh}(\mathcal{S}) = \lim_{n\to\infty} \ - \min_{U_0^{n-1}(\cdot) \ \text{s.t.} \ U[i](\cdot)\in\sigma(Y_0^n)} \frac{1}{n}\mathbb{E}\left[\log\left|\frac{X[n]}{X[0]}\right|\right].$$

We also call this the zeroth-moment control capacity of the system, since it essentially captures the weakest notion of system stability.

Theorem 5.1: Consider the system $\widetilde{\mathcal{S}}^a(p_B)$ (eq. (9)) and the affiliated system $\mathcal{S}(p_B)$ (eq. (8)), with identical initial conditions $X[0] = \widetilde{X}[0] \neq 0$, such that $-\infty < \mathbb{E}[\log|X[0]|] < \infty$, and identical $B[i]$'s.

Then $\exists \widetilde{U}_0^\infty(\cdot)$, such that $\exists M, N < \infty$, $\mathbb{E}[\log|\widetilde{X}[n]|] < M$ for all $n > N$, i.e the system is logarithmically stabilizable if and only if $C_{sh}(\mathcal{S}(p_B)) \geq \log|a|$.

The proof is omitted due to space constraints but is given in [14].

A. Calculating Shannon control capacity

Theorem 5.2: The Shannon control capacity of the system $\mathcal{S}(p_B)$ (eq. (8)), with i.i.d. $B[n]$'s, is ∞ if $p_B(\cdot)$ has an atom not at 0.

Theorem 5.3: The Shannon control capacity of the system $\mathcal{S}(p_B)$ (eq. (8)), where $p_B(\cdot)$ is a continuous distribution with no atoms is given by,

$$\max_{k\in\mathbb{R}} \mathbb{E}\left[-\log|1 + B \cdot k|\right] \tag{14}$$

where $B \sim p_B(\cdot)$.

Remark 5.1: Sadly, we must omit the proof due to space constraints, but it is available in [14]. Once again, as a consequence of the proof we see that linear strategies are optimal for calculating the Shannon control capacity. As in the zero-error case, this immediately gives us what we want even for the case of additive disturbances. This is because when the Shannon control-capacity is $> \log(a)$, the typical LTV sample paths realized by the closed-loop system are decaying exponentially and are hence BIBO stable.

Remark 5.2: To compute the Shannon control capacity of distributions $p_B(\cdot)$ with atoms at zero, we can remove the atom at zero and renormalize the result as $P(B \neq 0) \cdot C_{sh}\left(\mathcal{S}(p_{B^*}(\cdot))\right)$, where p_{B^*} is the renormalized distribution after removing the atom at 0.

VI. η-th moment control capacity

Finally, we can define the η-th moment stability. As η ranges from 0 to ∞ it captures a range of stabilities from the weakest "Shannon" sense as $\eta \to 0$, to the zero-error notions of stability as $\eta \to \infty$.

Definition 6.1: The system $\widetilde{\mathcal{S}}^a(p_{B[0]}, p_{B[1]}, \cdots)$ from (9) is said to be **η-th moment stabilizable** if there exists a causal control strategy $U_0^n(\cdot) \in \sigma(Y_0^n)$ such that $\exists M, N$ such that $M, N < \infty$ and $\mathbb{E}[|\widetilde{X}[n]|^\eta] < M$ for all $n > N$.

Definition 6.2: The η-th moment control capacity of the system $\mathcal{S}(p_{B[0]}, p_{B[1]}, \cdots)$ as in eq. (8) is

$$C_\eta(\mathcal{S}) = \liminf_{n\to\infty} \ - \min_{U_0^{n-1}(\cdot)\text{s.t.}U[i](\cdot)\in\sigma(Y_0^i)} \frac{1}{n}\ \frac{1}{\eta}\ \log\mathbb{E}\left[\left|\frac{X[n]}{X[0]}\right|^\eta\right].$$

Similar to the zero-error and Shannon notions of capacity we can state and prove a theorem regarding both the operational meaning of the η-th moment control capacity and give a single-letter expression for the η-th moment control capacity of the system $\mathcal{S}(p_B)$. These are available in [14].

Using this we can recover the classic uncertainty threshold principle result for unreliable actuation channels in system $\mathcal{S}(p_B)$ from (eq. (8)) as $C_2 = \frac{1}{2}\log\left(1 + \frac{\mu_B^2}{\sigma_B^2}\right)$.

Notice the form of the classical result. Because it is so reminiscent of the Shannon communication capacity of an AWGN channel, it was easy for the control community to miss that there were actually different senses of control capacity that needed to be developed.

VII. Examples

Here we calculate and plot the different notions of control capacity discussed in this paper. Fig. 2 plots the zero-error, Shannon and second-moment control capacities for a Gaussian distribution, a Bernoulli-(p) (erasure channel) distribution and a Uniform distribution. These distributions are normalized so

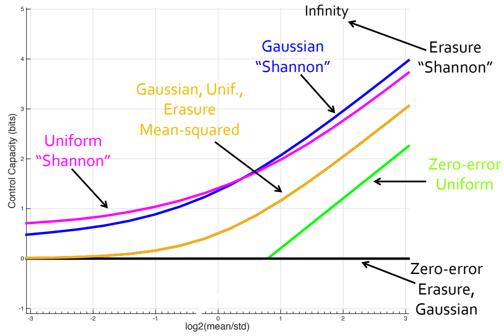

Fig. 2. This figure plots the various notions of control capacity for a Gaussian, a Bernoulli-(p) (erasure) and a Uniform distribution on the actuation channel. The x-axis plots the log of the ratio of the mean to the standard deviation for the channel distribution. This captures the fact that the ratio of the mean to the standard deviation is the only parameter that matters for the second-moment control capacity. The Shannon sense capacity for both the Gaussian and the uniform are larger than the second-moment capacity as expected. The Shannon capacity for the Bernoulli distribution is infinity since it contains an atom not at zero. The zero-error capacity for the Gaussian channel is zero because Gaussians are unbounded. The Bernoulli channel has no zero-error capacity because it has an atom at 0. The Uniform distribution is bounded and its zero-error capacity line has slope 1.

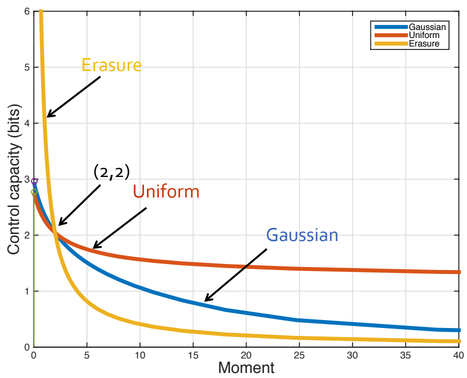

Fig. 3. The relationship between the different moment-senses of control capacity. For the Uniform, the zero-error control capacity is 1.2075 which is the asymptote as $\eta \to \infty$. As $\eta \to 0$ the η-th moment capacity converges to the Shannon sense. Here the Shannon control capacity for the Uniform is 2.7635 and for the Gaussian is 2.9586, which are the two small points seen on the extreme left (i.e. green and purple on the y-axis).

that they all have the same ratio of the mean to standard deviation (x-axis). The second moment capacities for all three line up exactly given the parameterization of the x-axis as expected.

Notice that as the ratio of mean to standard deviation goes to infinity, all of the lines approach slope 1. We conjecture that in this "high SNR" regime, this ratio is essentially what dictates the scaling of control capacity. This is predicted by the carry-free models since the capacity in both the zero-error and Shannon senses depends only on the number of deterministic bits in the control channel gain $g_{det} - g_{ran}$.

Fig. 3 explores the behavior of η-th moment capacities for the same three channels. The plot presents the η-th moment capacities for Gaussian, Erasure and Uniform control channels. We chose the three distributions such that their second-moment capacity is 2, and all three curves intersect there. As $\eta \to 0$, the curves approach the Shannon capacities and as $\eta \to \infty$ the curves asymptote at the zero-error control capacity. The results in this paper help us characterize this entire space, while previously only the $(2, 2)$ point was really known.

In future work, we clearly need to explore how these notions will extend in the case of vector systems.

VIII. Acknowledgements

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