Vector control systems with unpredictable actuation

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***Abstract*—Vector systems are fundamentally different from scalar systems since the dynamics of a vector system can both spin and stretch the system state. The magnitude and direction of control actions for multi-dimensional systems can need to be coordinated across time to achieve the desired behavior. Hence, planning for future actions is important. Consequently, system predictability, and the related ability to plan, can affect system stabilizability.**

This paper considers a simple d-dimensional state-space model for an unpredictable "spinning" control system. The spinning controller has a particular restriction: it may only apply a scalar control in a direction that is randomly chosen by nature at each time. This model is related to the classically studied models for intermittent Kalman filtering and intermittent control problems, but is simpler to analyze. The simplicity of the spinning model lets us use dynamic programming to characterize the decay rate of the system and thus understand the fundamental limitations imposed by the fact that future control directions are unpredictable. We can reduce the unpredictability of the model by providing non-causal side information about the future control directions, and further calculate how this affects the decay rate of the system. With k-steps of lookahead on the random control direction, the d-dimensional system decays as fast as a $d-k$ dimensional system with no lookahead on the control action. This model is a first step to understand unpredictability in more general networked control systems.

I. Introduction

The advent of networked control systems and the Internet of Things has made it important to understand stochastic models of parameter uncertainty in control, especially from an "information" point of view. For instance, previous work has extensively explored the effect of dropped observation packets [1], [2] as well as dropped control packets [3], [4] in control systems. Packet drops are one example of multiplicative uncertainty and unpredictability in a control system. This paper explores systems with unpredictable actuation.

Some previous work has explored multiplicative uncertainty in both the observation and actuation in systems. Multiplicative uncertainty in scalar systems puts many bottlenecks on our ability to control and estimate systems. In [5] we showed that an unstable system with observations that were received through a non-coherent channel cannot actually be tracked. However, a linear feedback control strategy can actually stabilize the system in certain regimes. Recently, we formulated a notion of control capacity to understand the limits on stabilization of a scalar system with multiplicative uncertainty on the actuation channel [6]. That paper considered the stabilization of a control system of the form $X[n+1] = a \cdot X[n] + B[n] \cdot U[n]$, where $U[n]$ was the control signal chosen based on a perfect observation of the state and $B[n]$'s were i.i.d. random variables unknown to the controller and freshly drawn at each time. Here, we provided an information-theoretic perspective on the classical uncertainty threshold principle result [7], and also extended this to notions beyond second-moment stability.

The control capacity formulation allows us to further capture the impact of side-information on the stabilizability of the system — we can characterize how much the control capacity of an actuation channel might increase if side-information about $B[n]$ is provided *by* time n. It is interesting to note that if $B[n]$ were revealed completely to the controller at time n, the control capacity of the actuation channel would be infinite — the state could be instantly set to zero. However, in this scalar formulation, side-information regarding $B[n]$ at time $n-1$ offers no increase in the control capacity over the same side-information received at time n. The scalar setting decouples the different time steps of the control problem — in this setting the ability to plan is not of value in terms of stabilizability. In fact, for the setting above a locally greedy strategy is optimal.

Vector systems exhibit different behavior. The essence of the problem is demonstrated by the following $d-$dimensional control system with state $\mathbf{X}[n]$ and with an unpredictable actuation vector $\mathbf{B}[n]$.

$$\begin{aligned}\mathbf{X}[n+1] &= \mathbf{X}[n] + \mathbf{B}[n]U[n] \\ \mathbf{Y}[n] &= \mathbf{X}[n]. \end{aligned} \tag{1}$$

The scalar control action $U[n]$ is what fundamentally couples the modes of the system and makes this a truly vector system as opposed to a stacked set of scalar systems. $\mathbf{B}[n]$ is the random actuation direction, and this is modeling the unpredictability in the system. We assume this is a vector drawn uniformly from the $d-$dimensional unit hypersphere [8].

This toy example illustrates how vector systems are fundamentally different from scalar systems. First, consider the uncertainty introduced by the actuation channel $\mathbf{B}[n]$. Say $\mathbf{B}[n]$ is randomly chosen, but perfectly revealed to the controller at time n. Even with perfect information about the actuation direction, the controller may not be able to immediately set the system state to the zero vector, $\mathbf{0}$, unlike the scalar system.

Further, since the different modes of the system are coupled, the ability to plan future controls is intimately connected to stabilizability. In fact, in the toy model above, having information about $\mathbf{B}[n]$ at time $n-1$ can change the stabilizability of the system.

Non-causal information in control has been investigated before. For instance, Davis observes in [9], that future side information about noise realizations can effectively reduce a stochastic control problem to a set of deterministic ones. Non-anticipative control and preview information have also been investigated [10], [11], [12].

In this paper, we investigate this set of toy vector control problems to understand the impact of unpredictability and the value of planning. The simple toy model allows us to provide the controller with limited amounts of lookahead on the randomness in the control direction and thus characterize the change in decay rate due to the lookahead. This paper builds on recently submitted work, where we discuss the value of side information in scalar systems [13]. The work also extends the problem setup explored in [14] from a limited two-dimensional setting to a $d-$dimensional setting and thus more fully explores the differences between the vector and the scalar systems. The "capacity" interpretation to the decay rates is justified in this paper by considering systems with intrinsic dynamics that would cause unstable growth in the absence of control.

II. Problem setup

We consider the following system as a first step to understand vector actuation channels.

$$\begin{aligned} \mathbf{X}[n+1] &= \mathbf{X}[n] + \mathbf{B}[n]U[n] \\ \mathbf{Y}[n] &= \mathbf{X}[n] \end{aligned} \tag{2}$$

$\mathbf{X}[n]$ is the d dimensional system state. The system state is perfectly observed through $\mathbf{Y}[n]$. $U[n]$ is the scalar control that is chosen. The scalar control action is what fundamentally couples the modes of the system and makes this a truly vector system as opposed to a stacked set of scalar systems. $\mathbf{B}[n]$ is the random actuation direction — what we are using to model the unpredictability in the system. We assume this is a random vector drawn on the $d-$dimensional unit hypersphere. The vector $B[n]$ maybe be generated by drawing the components from a standard Normal i.i.d. and then normalizing the vector to be on the unit hypersphere. Alternatively, it can be generated by taking a standard basis unit vector and multiplying it by a random rotation matrix chosen uniformly from $\text{SO}(d)$.

In Sec. V, we will consider the same system with a growth matrix $\mathbf{A}$ and also a rotation matrix, $\Phi[n]$, that hits the system state in between control actions. $\Phi[n]$ is chosen uniformly at random from $\text{SO}(d)$ using an algorithm as that in [8]. The results for Sec. III and Sec. IV do not require such a rotation $\Phi[n]$.

The scalar control $U[n]$ is chosen by the controller. The controller is interested in minimizing the magnitude of the system state after N timesteps. $U[n]$ at time n can depend on any current and past observations $\mathbf{Y}[0], \ldots, \mathbf{Y}[n]$, and also on any side-information that might be provided to it.

If we consider the setup where the actuation direction $\mathbf{B}[n]$ is completely unknown to the controller at time n. In this case, it is clear that the optimal control action is $U[n] = 0$. The system state would not decay at all and the "control capacity" as we might intuitively understand it should be zero.

In this paper, we consider setups where we have more information about the actuation channel $\mathbf{B}[n]$. First, in Sec. III the current actuation direction $\mathbf{B}[n]$ is provided to he controller at time n. Sec. IV considers a setup where the directions $\mathbf{B}[n], \ldots, \mathbf{B}[n+k]$ are all provided to the controller. Finally, Sec. V introduces a notion of intrinsic system growth to the problem.

As a measure of performance, we are interested in the decay rate of the system,

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2}. \tag{3}$$

We will find that this value is independent of $\mathbf{x}[0]$. This is motivated by our desire to understand a notion of "control capacity" for the vector actuation. Building on the definition in [6], we define the 2nd-moment control capacity for the system $\mathcal{S}$:

Definition 2.1: The 2-nd moment control capacity, $C_2(\mathcal{S})$ of the $d-$dimentional system $\mathcal{S}$ as in eq. (2) with initial state $\mathbf{x}[0]$ is defined as

$$\liminf_{N\to\infty} \quad -\min_{U_0^{N-1}(\cdot) \text{s.t.} U[n](\cdot)\in\sigma(\mathbf{Y}[0],\ldots,\mathbf{Y}[n])\forall n} \; d\frac{1}{N}\;\frac{1}{2}\;\log \mathbb{E}\left[\frac{||\mathbf{X}[N]||^2}{||\mathbf{x}[0]||^2}\right].$$

This definition parallels the definition in [6]. However, it accounts for the fact that a vector system has to control growth in $d-$dimensions, which gives the extra factor of d outside the log. We believe that this dimension factor plays a role similar to "bandwidth" in wireless communication.

The following notation will be useful in the rest of the paper. Let $\mathbb{S}_n$ be the set that represents the actuation directions that are known at time n. This is related to the classical notion of "information patterns" in control, except these span time instead of contemporary states. Different variations on the problem will involve augmenting this set in the later parts of the paper. For the system with no lookahead $\mathbb{S}_n = \{\mathbf{B}[n]\}$.

We will use a dynamic programming approach to calculate the optimal control strategy, and define our terminal cost as:

$$J_N(\mathbf{X}[N], \mathbb{S}_n) = \mathbb{E}[\|\mathbf{X}[N+1]\|^2] = \mathbb{E}[\|\mathbf{X}[N] + \mathbf{B}[N]U[N]\|^2]. \tag{4}$$

The expectation is taken over all randomness in the system. Let $J_n(\mathbf{X}[n], \mathbb{S}_n, U[n])$ be the cost-to-go at timestep n. The optimal control action at time n defines $J_n^*(\mathbf{X}[n], \mathbb{S}_n)$.

$$J_n^*(\mathbf{X}[n], \mathbb{S}_n) = \min_{U[n]} \mathbb{E}[J_n(\mathbf{X}[n], \mathbb{S}_n, U[n])]. \tag{5}$$

where $\mathbb{S}_n$ is the set of known vectors at timestep n, and the expectation is taken over future randomness in the system.

Let

$$\mathbf{x}^{\perp L} = \mathbf{x} - \text{proj}_L \mathbf{x} \tag{6}$$

be the orthogonal component of a vector $\mathbf{x}$ to some subspace (or vector) L. Also we define

$$\mathbf{x}^{\| L} = \text{proj}_L \mathbf{x} \tag{7}$$

to be the projection of a vector $\mathbf{x}$ onto some subspace (or vector) L.

The following folk ("least squares") lemma will let us interchange the notions of minimization and projection and is useful in proofs.

Lemma 2.1: Let $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be linearly independent vectors in $\mathbb{R}^d$.

$$\min_{u_1,\ldots,u_m} \|\mathbf{x} + u_1\mathbf{b}_1 + \cdots + u_m\mathbf{b}_m\|^2 = \|\mathbf{x}^{\perp \mathbf{b}_1,\ldots,\mathbf{b}_m}\|^2 \quad (8)$$

the expression is minimized by taking the residue of $\mathbf{x}$ from the subspace spanned by $\mathbf{b}_1, \ldots, \mathbf{b}_m$, with a unique solution on $u_1, \ldots, u_m$. The proof is omitted because of the page limit.

III. With No Lookahead/No Causal Side-Information

To be able to understand the optimal control for the problem we will repeatedly use the following key lemma that helps us understand how random projections affect norms.

Lemma 3.1: Let L be an ℓ-dimensional subspace of $\mathbb{R}^d$ that is chosen uniformly at random over d-space, and let $\mathbf{x}$ be any arbitrary vector in d-space, then

$$\mathbb{E}_L[\|\mathbf{x}^{\|L}\|^2] = \frac{\ell}{d}\|\mathbf{x}\|^2. \quad (9)$$

Proof: Assume that the subspace L of dimension ℓ is chosen uniformly at random using the following procedure[1]. Choose ℓ unit vectors at random and then perform the Gram-Schmidt process, as done in [8]. This will give us a set of ℓ orthonormal vectors that will almost surely span an $\ell-$dimensional subspace of $\mathbb{R}^d$.

Once we have a basis for L, we can extend the basis to the whole space $\mathbb{R}^d$ by choosing $d-\ell$ more orthonormal vectors in the same way we chose the first ℓ basis vectors. We will call this orthonormal basis $\{\mathbf{E}_1, \ldots, \mathbf{E}_d\}$. The first ℓ vectors will be the orthonormal basis for the subspace L. Then we can express the norm of vector $\mathbf{x}$ as:

$$\|\mathbf{x}\|^2 = \|\mathbf{x}^{\|\mathbf{E}_1}\|^2 + \cdots + \|\mathbf{x}^{\|\mathbf{E}_d}\|^2. \quad (10)$$

We take the expectation of both sides, over $\{\mathbf{E}_1, \ldots, \mathbf{E}_d\}$.

$$\begin{aligned}
&\mathbb{E}_{\{\mathbf{E}_1,\ldots,\mathbf{E}_d\}}[\|\mathbf{x}\|^2] \\
&= \mathbb{E}_{\{\mathbf{E}_1,\ldots,\mathbf{E}_d\}}[\|\mathbf{x}^{\|\mathbf{E}_1}\|^2 + \cdots + \|\mathbf{x}^{\|\mathbf{E}_d}\|^2] \\
&= \mathbb{E}_{\{\mathbf{E}_1,\ldots,\mathbf{E}_d\}}[\|\mathbf{x}^{\|\mathbf{E}_1}\|^2] + \cdots + \mathbb{E}_{\{\mathbf{E}_1,\ldots,\mathbf{E}_d\}}[\|\mathbf{x}^{\|\mathbf{E}_d}\|^2] \\
&= \mathbb{E}_{\mathbf{E}_1}[\|\mathbf{x}^{\|\mathbf{E}_1}\|^2] + \cdots + \mathbb{E}_{\mathbf{E}_d}[\|\mathbf{x}^{\|\mathbf{E}_d}\|^2], \quad (11)
\end{aligned}$$

by the linearity of expectation.

Since the random basis vectors are symmetric, all the terms on the right hand side are equal.

$$\mathbb{E}_{\mathbf{E}_i}[\|\mathbf{x}^{\|\mathbf{E}_i}\|^2] = \mathbb{E}_{\mathbf{E}_j}[\|\mathbf{x}^{\|\mathbf{E}_j}\|^2]. \quad (12)$$

[1] An alternative way to choose a subspace would be to choose the first basis vector for the subspace uniformly at random and the subsequently choose the remaining basis vectors from the space orthogonal to the vectors already chosen. This is equivalent to the other strategy.

Also $\mathbb{E}_{\{\mathbf{E}_1,\ldots,\mathbf{E}_d\}}[\|\mathbf{x}\|^2] = \|\mathbf{x}\|^2$, since $\mathbf{x}$ is a fixed vector. Hence, we have:

$$\mathbb{E}_{\mathbf{E}_i}[\|\mathbf{x}^{\|\mathbf{E}_i}\|^2] = \frac{1}{d}\|\mathbf{x}\|^2. \quad (13)$$

Since L is the subspace spanned by the first ℓ vectors, by the linearity of expectations,

$$\begin{aligned}
\mathbb{E}_L[\|\mathbf{x}^{\|L}\|^2] &= \mathbb{E}_{\{\mathbf{E}_1,\ldots,\mathbf{E}_\ell\}}[\|\mathbf{x}^{\|\mathbf{E}_1}\|^2 + \cdots + \|\mathbf{x}^{\|\mathbf{E}_l}\|^2] \\
&= \mathbb{E}_{\mathbf{E}_1}[\|\mathbf{x}^{\|\mathbf{E}_1}\|^2] + \cdots + \mathbb{E}_{\mathbf{E}_\ell}[\|\mathbf{x}^{\|\mathbf{E}_\ell}\|^2] \\
&= \frac{\ell}{d}\|\mathbf{x}\|^2. \quad (14)
\end{aligned}$$

■

A second lemma that deals with projections of random vectors onto fixed subspaces also captures the same idea.

Lemma 3.2: Let $\mathbf{X}$ be a unit norm vector chosen uniformly at random in $\mathbb{R}^d$. Let L be any fixed arbitrary ℓ dimensional subspace of $\mathbb{R}^d$:

$$\mathbb{E}_{\mathbf{X}}[\|\mathbf{X}^{\|L}\|^2] = \frac{\ell}{d} \quad (15)$$

Proof:

The proof is very similar. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ be an orthonormal basis for $\mathbb{R}^d$ such that the first ℓ vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_\ell\}$ form a basis for L. Then,

$$\|\mathbf{X}\|^2 = \|\mathbf{X}^{\|\mathbf{e}_1}\|^2 + \cdots + \|\mathbf{X}^{\|\mathbf{e}_d}\|^2. \quad (16)$$

We take the expectation on both sides, over $\mathbf{X}$.

$$\begin{aligned}
\mathbb{E}_{\mathbf{X}}[\|\mathbf{X}\|^2] &= \mathbb{E}_{\mathbf{X}}[\|\mathbf{X}^{\|\mathbf{e}_1}\|^2 + \cdots + \|\mathbf{X}^{\|\mathbf{e}_d}\|^2] \quad (17) \\
1 &= \mathbb{E}_{\mathbf{X}}[\|\mathbf{X}^{\|\mathbf{e}_1}\|^2] + \cdots + \mathbb{E}_{\mathbf{X}}[\|\mathbf{X}^{\|\mathbf{e}_d}\|^2]. \quad (18)
\end{aligned}$$

Again by symmetry,

$$\mathbb{E}_{\mathbf{X}}[\|\mathbf{X}^{\|\mathbf{e}_i}\|^2] = \mathbb{E}_{\mathbf{X}}[\|\mathbf{X}^{\|\mathbf{e}_j}\|^2]. \quad (19)$$

Therefore,

$$\mathbb{E}_{\mathbf{X}}[\|\mathbf{X}^{\|\mathbf{e}_i}\|^2] = \frac{1}{d}. \quad (20)$$

Now if we only consider the ℓ dimensions of the subspace L

$$\begin{aligned}
\mathbb{E}_X[\|\mathbf{X}^{\|L}\|^2] &= \mathbb{E}_X[\|\mathbf{X}^{\|\mathbf{e}_1}\|^2 + \cdots + \|\mathbf{X}^{\|\mathbf{e}_\ell}\|^2] \quad (21) \\
&= \mathbb{E}_X[\|\mathbf{X}^{\|\mathbf{e}_1}\|^2] + \cdots + \mathbb{E}_X[\|\mathbf{X}^{\|\mathbf{e}_\ell}\|^2] \quad (22) \\
&= \frac{\ell}{d} \quad (23)
\end{aligned}$$

■

Now, we can state the primary theorem about decay rate for the basic system with no lookahead.

Theorem 3.3: For the system with no lookahead as in eq. (2) the optimal strategy control action at time n is

$$U[n] = -\mathbf{X}[n]^{\|\mathbf{B}[n]} \quad (24)$$

with a decay given by

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} = \left(\frac{d-1}{d}\right)^N. \quad (25)$$

We prove this by using dynamic programming to obtain the optimal policy. We ignore the rotation matrix $\Phi[n]$ that rotates the axes between time steps for ease of exposition — this does not affect the results in any way and will play a greater role in the section of the paper that has intrinsic dynamics. The proof of this builds on the following lemma that captures the induction step.

Lemma 3.4: For $0 \leq n \leq N-1$

$$J_n^*(\mathbf{X}[n], \mathbf{B}[n]) = \left(\frac{d-1}{d}\right)^{N-n-1} \|\mathbf{X}[n]^{\perp \mathbf{B}[n]}\|^2. \quad (26)$$

Proof: We prove this using backwards induction. Clearly, this is true for $n = N-1$, since with only one action to be taken the optimal control strategy is to remove the component of $\mathbf{X}[N]$ along $\mathbf{B}[N]$ which is the only direction that the control can act in. Hence, $J_N^*(\mathbf{X}[N], \mathbf{B}[N]) = \|\mathbf{X}[N]^{\perp \mathbf{B}[N]}\|^2$.

Now, let us suppose that our statement is true for $n+1$. Then, we can compute the optimal cost-to-go at time n in terms of the cost at time $n+1$:

$$J_n^*(\mathbf{X}[n], \mathbf{B}[n]) \quad (27)$$

$$= \min_{U[n]} \mathbb{E}_{\mathbf{B}[n+1]}[J_{n+1}^*(\mathbf{X}[n+1], \mathbf{B}[n+1])] \quad (28)$$

$$= \min_{U[n]} \mathbb{E}_{\mathbf{B}[n+1]}[\left(\frac{d-1}{d}\right)^{N-n-2} \|\mathbf{X}[n+1]^{\perp \mathbf{B}[n+1]}\|^2] \quad (29)$$

$$= \left(\frac{d-1}{d}\right)^{N-n-2} \min_{U[n]} \mathbb{E}_{\mathbf{B}[n+1]}[\|\mathbf{X}[n+1]^{\perp \mathbf{B}[n+1]}\|^2]. \quad (30)$$

Now, by the Lemma 3.1 proved earlier, since $\mathbf{B}[n+1]$ is a random vector and hence the space orthogonal to $\mathbf{B}[n+1]$ is a $d-1$ dimensional random subspace:

$$\mathbb{E}_{\mathbf{B}[n+1]}[\|\mathbf{X}[n+1]^{\perp \mathbf{B}[n+1]}\|^2] = \left(\frac{d-1}{d}\right) \|\mathbf{X}[n+1]\|^2.$$

$$J_n^*(\mathbf{X}[n], \mathbf{B}[n])$$

$$= \left(\frac{d-1}{d}\right)^{N-n-2} \min_{U[n]} \left(\frac{d-1}{d}\right) \|\mathbf{X}[n+1]\|^2 \quad (31)$$

$$= \left(\frac{d-1}{d}\right)^{N-n-1} \min_{U[n]} \|\mathbf{X}[n+1]\|^2 \quad (32)$$

$$= \left(\frac{d-1}{d}\right)^{N-n-1} \|\mathbf{X}[n]^{\perp \mathbf{B}[n]}\|^2 \quad (33)$$

The last step follows because of the minimizing control $U[N]$ is to project the state $\mathbf{X}[n]$ onto the actuation direction $\mathbf{B}[n]$, by Lemma 2.1. ■

Now we can prove Thm. 3.3.

Proof: The system objective is to minimize the terminal cost, $J_N(\mathbf{X}[N])$. Then, for the performance,

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} = \frac{\mathbb{E}_{\mathbf{B}[0]}[J_0^*(\mathbf{x}[0], \mathbf{B}[0])]}{\|\mathbf{x}[0]\|^2} \quad (34)$$

$$= \left(\frac{d-1}{d}\right)^{N-1} \frac{\mathbb{E}_{\mathbf{B}[0]}[\|\mathbf{x}[0]^{\perp \mathbf{B}[0]}\|^2]}{\|\mathbf{x}[0]\|^2} \quad (35)$$

By Lemma 3.1,

$$\mathbb{E}_{\mathbf{B}[0]}[\|\mathbf{x}[0]^{\perp \mathbf{B}[0]}\|^2 = \frac{d-1}{d} \|\mathbf{x}[0]\|^2 \quad (36)$$

Therefore,

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} = \left(\frac{d-1}{d}\right)^{N} \quad (37)$$

■

Corollary 3.5: The second-moment control capacity of the system with no-lookahead as in eq. (2) is $\frac{d}{2} \log\left(1 + \frac{1}{d-1}\right)$.

It is interesting to consider the asymptotic limit of this as $d \to \infty$. It becomes $\frac{1}{2}$ nats (if we take natural logs) and $\frac{1}{2 \ln 2}$ bits. This is reminiscent of results in the wideband limit for communication over an AWGN channel, where one gets an equivalence between joules and bits.

IV. With Lookahead/Causal Side-Information

We would like to now understand how our ability to stabilize the system increases as we make the actuation more predictable by reducing the uncertainty through explicit information. In particular, we would like to understand the case where the controller knows the next k actuation directions in addition to the current actuation direction at time n.

The effect of non-causal side information in systems was partially explored in [14], [15] for the 2-dimensional case. There, we saw that lookahead about the actuation direction for one step in the future can improve the decay rate of the system by a factor of infinity in the two dimensional case.

With k steps of lookahead, the set of actuation directions known at time n becomes:

$$\mathbb{S}_n = \{\mathbf{B}[n], \ldots, \mathbf{B}[n+k]\} \quad (38)$$

Let $L_n = \text{span } \mathbb{S}_n$.

Theorem 4.1: Consider the system in eq. (2), where the controller is given the set of vectors $\mathbb{S}_n$ as side information at time n. Then, if

$$\mathbf{X}[n]^{\| L_n} = u_{n,n}\mathbf{B}[n] + \cdots + u_{n,n+k}\mathbf{B}[n+k] \quad (39)$$

the optimal control action at time n is given by $U[n] = -u_{n,n}$, with a performance of

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} = \frac{d-k-1}{d} \left(\frac{d-k-1}{d-k}\right)^{N-k-1}. \quad (40)$$

We note that since $\mathbf{B}[n], \ldots, \mathbf{B}[n+k]$ are almost surely linearly independent, there is a unique set of $u_{n,i}, n \leq i \leq n+k$ that satisfy Eq. (39). The result here has a spirit similar to "receding-horizon control" in that we compute hypothetical controls to apply for the next k time steps but only apply the first one.

Proof: We prove this by using dynamic programming to obtain the optimal policy. Again, the system objective is to minimize the terminal cost, $J_N(\mathbf{X}[N]) = \mathbb{E}[\|\mathbf{X}[N]\|^2]$.

Let $J_n(\mathbf{X}[n], \mathbb{S}_n, U[n])$ be the cost incurred at timestep n in state $\mathbf{X}[n]$ given the lookahead about actuation directions in $\mathbb{S}_n$. $U[n]$ is the scalar control applied.

Then for $n \leq N-k-1$,

$$J_n(\mathbf{X}[n], \mathbb{S}_n, U[n]) = \mathbb{E}_{\mathbf{B}[n+k+1]}[J^*_{n+1}(\mathbf{X}[n+1], \mathbb{S}_{n+1})].$$

Similar to the case with no-lookahead, we capture the backwards induction step in a lemma to capture the factor $\left(\frac{d-k-1}{d-k}\right)^{N-k-1}$.

Lemma 4.2: For $0 \leq n \leq N-k-1$,

$$J^*_n(\mathbf{X}[n], \mathbb{S}_n) = \left(\frac{d-k-1}{d-k}\right)^{N-n-k-1} \|\mathbf{X}[n]^{\perp L_n}\|^2 \quad (41)$$

Proof: We note that the statement is true for $n = N-k-1$ as the base case. This follows from Lemma 2.1. The optimal controls $U[N-k-1], \ldots, U[N-1]$ can be determined at time $n = N-k-1$, and the state magnitude at time N will be given by the error when the state is projected onto the subspace spanned by the actuation directions.

Then, assume the statement is true for the case $n+1$. Then we can apply the induction hypothesis to the cost-to-go at time n to get the following:

$$\begin{aligned} &J^*_n(\mathbf{X}[n], \mathbb{S}_n) \\ &= \min_{U[n]} \mathbb{E}_{B[n+k+1]}[J^*_{n+1}(\mathbf{X}[n+1], \mathbb{S}_{n+1})] \\ &= \min_{U[n]} \mathbb{E}_{\mathbf{B}[n+k+1]}\left[\left(\frac{d-k-1}{d-k}\right)^{N-n-k-2} \|\mathbf{X}[n+1]^{\perp L_{n+1}}\|^2\right] \end{aligned}$$

To complete the proof of the lemma, we only need to consider

$$\min_{U[n]} \mathbb{E}_{\mathbf{B}[n+k+1]}[\|\mathbf{X}[n+1]^{\perp L_{n+1}}\|^2] \quad (42)$$

Note that $\mathbf{B}[n+k+1] \in L_{n+1}$. So it is easier to understand the quantity above when divide the subspace L_{n+1} into the space spanned by $\mathbf{B}[n+k+1]$ and the remaining dimensions.

So now consider the set of the first k vectors in $\mathbb{S}_{n+1}$

$$\mathbb{T}_{n+1} = \{\mathbf{B}[n+1], \ldots, \mathbf{B}[n+k]\} \quad (43)$$

and its span $\mathcal{T}_n = \text{span } \mathbb{T}_n$. We note that

$$\mathcal{T}_{n+1} \oplus \mathbf{B}[n+k+1]^{\perp \mathcal{T}_{n+1}} = L_{n+1} \quad (44)$$

The following lemma, which we prove in the Appendix (Sec. VII), helps to understand this. This basically follows from the linearity of projection.

Lemma 4.3: Let $G_1, G_2 \subset G$ be subspaces such that $G_1 \oplus G_2 = G$. Then,

$$\mathbf{x}^{\perp G} = (\mathbf{x}^{\perp G_1})^{\perp G_2} \quad (45)$$

Applying this Lemma and using eq. (44) we have

$$\mathbf{X}[n+1]^{\perp L_{n+1}} = (\mathbf{X}[n+1]^{\perp \mathcal{T}_{n+1}})^{\perp (\mathbf{B}[n+k+1]^{\perp \mathcal{T}_{n+1}})}.$$

$\mathbf{X}[n+1]^{\perp \mathcal{T}_{n+1}}$ is a $d-k$ dimensional vector, and $\mathbf{B}[n+k+1]^{\perp \mathcal{T}_{n+1}}$ is a 1-dimensional vector chosen uniformly at random in that space. Hence, by Lemma. 3.1,

$$\begin{aligned} &\mathbb{E}_{\mathbf{B}[n+k+1]}[\|(\mathbf{X}[n+1]^{\perp \mathcal{T}_{n+1}})^{\perp (\mathbf{B}[n+k+1]^{\perp \mathcal{T}_{n+1}})}\|^2] \quad (46) \\ &= \left(\frac{d-k-1}{d-k}\right) \|\mathbf{X}[n+1]^{\perp \mathcal{T}_{n+1}}\|^2 \quad (47) \end{aligned}$$

Now we know that

$$\begin{aligned} &\min_{U[n]} \|\mathbf{X}[n+1]^{\perp \mathcal{T}_{n+1}}\|^2 \\ &= \min_{U[n]}(\min_{U[n+1],\ldots,U[n+k]} \|\mathbf{X}[n+k+1]\|^2) (\text{by Lemma 2.1}) \\ &= \min_{U[n],\ldots,U[n+k]} \|\mathbf{X}[n+k+1]\|^2 \quad (48) \\ &= \|\mathbf{X}[n]^{\perp L_n}\|^2 \quad (49) \end{aligned}$$

The last step again follows by Lemma. 2.1.

Therefore,

$$J^*_n(\mathbf{X}[n], \mathbb{S}_n) = \left(\frac{D-k-1}{D-k}\right)^{N-n-k-1} \|\mathbf{X}[n]^{\perp L_n}\|^2] \quad (50)$$

This proves the lemma. ■

Continuing the proof of the theorem, we have:

$$\begin{aligned} \frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} &= \frac{\mathbb{E}_{\mathbb{S}_0}[J^*_0(\mathbf{x}[0], \mathbb{S}_0)]}{\|\mathbf{x}[0]\|^2} \quad (51) \\ &= \left(\frac{d-k-1}{d-k}\right)^{N-k-1} \frac{\mathbb{E}_{\mathbb{S}_0}[\|\mathbf{x}[0]^{\perp L_0}\|^2]}{\|\mathbf{x}[0]\|^2} \quad (52) \end{aligned}$$

By Lemma 3.1,

$$\mathbb{E}_{\mathbb{S}_0}[\|\mathbf{x}[0]^{\perp L_0}\|^2] = \frac{d-k-1}{d}\|\mathbf{x}[0]\|^2 \quad (53)$$

Therefore,

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} = \frac{d-k-1}{d}\left(\frac{d-k-1}{d-k}\right)^{N-k-1} \quad (54)$$

■

With this, we see that the control capacity increases with side information about future realizations of the actuation channel, and in fact in terms of control capacity, one step of lookahead is equivalent to reducing the dimensionality of the ambient problem space

Corollary 4.4: The second-moment control capacity of the system, as in eq. (2), with k-steps of lookahead where k is finite, is $\frac{d}{2}\log\left(\frac{d-k}{d-k-1}\right)$.

It is interesting again to take the limits as $d \to \infty$. Once again, we get the second moment control capacity approaching $\frac{1}{2}$ nats, regardless of how big k is. If we allow the lookahead k to scale with d, say $k = \alpha d$ for some $0 < \alpha < 1$, then the asymptotic second moment control capacity approaches $\frac{1}{2(1-\alpha)}$ nats. It grows unboundedly as α approaches 1.

V. Connections to system growth rate

Finally, we would like to connect the notions of vector control capacity to system growth, so in this section we consider a system where the system state grows by the matrix A at each time step and investigate the stabilizability of this system. We will consider the case where A is a real-valued, full-rank, symmetric $d \times d$ matrix.

$$\begin{aligned} \mathbf{X}[n+1] &= \Phi[n+1]\mathbf{X}'[n+1] \\ \mathbf{X}'[n+1] &= \mathbf{A}\mathbf{X}[n] + U[n]\mathbf{B}[n] \quad (55) \\ \mathbf{Y}[n] &= \mathbf{X}[n] \end{aligned}$$

In addition to the system growth by the factor $\mathbf{A}$, the system state is hit by a random rotation matrix $\Phi[n+1]$ after the control at time n has been applied. $\Phi[n]$. This rotation matrix is chosen uniformly at random from the group SO(d). It operates to randomize the direction of the state vector between control actions at every time step.

For this section, let $\mathbf{x}[0]$ be the initial state. It is also rotated uniformly at random in the first step, so $\mathbf{X}[0] = \Phi[0]\mathbf{x}[0]$.

The matrix $\mathbf{A}$ is fixed and known to the controller. To be careful, we define the lookahead information that is available to the controller. For time $n_1 < n$, for a vector $\mathbf{B}[n]$, let $\mathbf{B}[n]$ as "seen at time n_1" be

$$\mathbf{B}_{n_1}[n] = \Phi[n_1+1]^{-1}\mathbf{A}^{-1}\cdots\Phi[n]^{-1}\mathbf{A}^{-1}\mathbf{B}[n].$$

Clearly, $\mathbf{B}_n[n] = \mathbf{B}[n]$. Denote the normalized versions of these vectors as $\mathbf{B}'_{n_1}[n] = \frac{\mathbf{B}_{n_1[n]}}{||\mathbf{B}_{n_1[n]}||^2}$.

At time n the controller receives only the normalized directions[2] as lookahead

$$\mathbb{S}_n = \{\mathbf{B}'_n[n], \ldots, \mathbf{B}'_n[n+k]\} \tag{56}$$

Let $L_n = \text{span } \mathbb{S}_n$. Also we define

$$\mathbb{T}_n = \{\mathbf{B}'_n[n], \ldots, \mathbf{B}'_n[n+k-1]\} \tag{57}$$

and $\mathcal{T}_n = \text{span } \mathbb{T}_n$. Thus, at timestep n, you have perfect information about $\mathbf{X}[n]$ and $\mathbf{B}[n]$. Because the vectors are normalized and only the direction is provided to the controller, no information about any of the rotation matrices $\Phi[n]$ is ever leaked to the controller. This is because controller only ever view's the $\Phi[\cdot]$ matrices projected in a random direction. At each timestep, one layer of $\Phi^{-1}A^{-1}$ is "peeled off," but it is only possible for the controller to learn anything about $\Phi[n]$ after it has already been applied to the system.

Analogously, we can define, for $n_1 < n$,

$$\mathbf{X}_{n_1}[n] = \mathbf{A}^{-1}\Phi[n_1+1]^{-1}\cdots\mathbf{A}^{-1}\Phi[n]^{-1}\mathbf{X}[n].$$

In the previous sections, $\mathbf{A}$ was effectively the identity matrix, and hence there was no issue of the scaling of different dimensions. The introduction of a general diagonal $\mathbf{A}$ means that different actuations directions will scale differently as time evolves. The rotating random $\Phi[n]$'s prevent any exploitation of this.

Before the main theorem we state a lemma that describes how the matrix $\mathbf{A}$ scales a random vector.

Lemma 5.1: For any $\mathbf{x}$, symmetric matrix $\mathbf{A}$, and a rotation matrix Φ drawn uniformly at random from SO(d),

$$\mathbb{E}_\Phi[\|\mathbf{A}\Phi\mathbf{x}\|^2] = \frac{\sum_{i=1}^d \lambda_i^2}{d}\|\mathbf{x}\|^2 \tag{58}$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $\mathbf{A}$.

[2]It is vitally important that the lookahead be adjusted to undo the effect of the spins and dynamics to come or we risk having the lookahead information be uninterpretably useless.

Proof: Since $\mathbf{A}$ is symmetric, let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the orthonormal basis of real eigenvectors corresponding to $\mathbf{A}$, and we can write, $\Phi\mathbf{x} = \sum_{i=1}^d (\Phi\mathbf{x})^{\|\mathbf{e}_i}$ By linearity,

$$A\Phi\mathbf{x} = \sum_{i=1}^d A(\Phi\mathbf{x})^{\|\mathbf{e}_i}. \tag{59}$$

Because $\mathbf{e}_i$ are eigenvectors,

$$\sum_{i=1}^d A(\Phi\mathbf{x})^{\|\mathbf{e}_i} = \sum_{i=1}^d \lambda_i(\Phi\mathbf{x})^{\|\mathbf{e}_i}. \tag{60}$$

Then, we take norms on both sides and because the vectors are orthogonal,

$$\|\sum_{i=1}^d \lambda_i(\Phi\mathbf{x})^{\|\mathbf{e}_i}\|^2 = \sum_{i=1}^d \lambda_i^2\|(\Phi\mathbf{x})^{\|\mathbf{e}_i}\|^2. \tag{61}$$

By the linearity of expectations,

$$\mathbb{E}_\Phi[\sum_{i=1}^d \lambda_i\|(\Phi\mathbf{x})^{\|\mathbf{e}_i}\|^2] = \sum_{i=1}^d \lambda_i^2\mathbb{E}_\Phi[\|(\Phi\mathbf{x})^{\|\mathbf{e}_i}\|^2]. \tag{62}$$

Now by Lemma 3.1, since $\Phi\mathbf{x}$ is a random vector, $\mathbb{E}_\Phi[\|(\Phi\mathbf{x})^{\|\mathbf{e}_i}\|^2] = \frac{1}{d}\|\mathbf{x}\|^2$. Therefore,

$$\sum_{i=1}^d \lambda_i^2\mathbb{E}_\Phi[\|(\Phi\mathbf{x})^{\|\mathbf{e}_i}\|^2] = \frac{\sum_{i=1}^d \lambda_i^2}{d}\|\mathbf{x}\|^2. \tag{63}$$

∎

Now we can state the main result of this section.

Theorem 5.2: Let

$$(A\mathbf{X}[n])^{\|\mathcal{S}_n} = u_{n,n}\mathbf{B}'_n[n] + u_{n,n+1}\mathbf{B}'_n[n+1] + \cdots + u_{n,n+k}\mathbf{B}'_n[n+k]$$

then, the optimal control action for the system in eq. (55) with k-steps of lookahead through $\mathbb{S}_n$ is given by $U[n] = -u_{n,n}$, with a performance of

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} = \frac{d-k-1}{d}\left(\frac{d-k-1}{d-k}\right)^{N-k-1}\left(\frac{\sum_{i=1}^d \lambda_i^2}{d}\right)^{N-1}$$

We state an induction Lemma before the proof of this theorem.

Lemma 5.3: For $n \leq N-k-1$,

$$J_n^*(\mathbf{X}[n], \mathbb{S}_n) = \left(\frac{\sum_{i=1}^d \lambda_i^2}{d}\right)^{N-n-2} \cdot \left(\frac{d-k-1}{d-k}\right)^{N-n-k-1}\|(A\mathbf{X}[n])^{\perp L_n}\|^2. \tag{64}$$

Proof: First, we look at the base case. For $n = N-k-1$,

$$\begin{aligned} &J_{N-k-1}^*(\mathbf{X}[N-k-1], \mathbb{S}_{N-k-1}) \\ &= \min_{U[N-k-1],\ldots,U[N-1]} \mathbb{E}_{\Phi[N-k],\ldots\Phi[N]}[\|\mathbf{X}_N[N]\|^2] \\ &= \min_{U[N-k-1],\ldots,U[N-1]} \mathbb{E}_{\Phi[N-k],\ldots,\Phi[N]}[\\ &\quad \|\Phi[N]A\cdots\Phi[N-k]A\mathbf{X}_{N-k-1}[N]\|^2]. \qquad (65) \\ &= \left(\frac{\sum_{i=1}^d \lambda_i^2}{d}\right)^{k-1} \min_{U[N-k-1],\ldots,U[N-1]} \|A\mathbf{X}_{N-k-1}[N]\|^2. \end{aligned}$$

by Lemma 5.1. And then by Lemma 2.1 we can rewrite this as:

$$\left(\frac{\sum_{i=1}^{d} \lambda_i^2}{d}\right)^{k-1} \|(A\mathbf{X}[N-k-1])^{\perp L_{N-k-1}}\|^2. \quad (66)$$

So the case $n = N-k-1$ is true. Next, we have that

$$J_n^*(\mathbf{X}[n], \mathbb{S}_n) = \min_{U[n]} \mathbb{E}_{\mathbf{B}[n+k+1],\Phi[n+1]}[J_{n+1}^*(\mathbf{X}[n+1], \mathbb{S}_{n+1})]$$

Applying the induction hypothesis, this equals

$$= \min_{U[n]} \mathbb{E}_{\mathbf{B}[n+k+1],\Phi[n+1]} \left[\left(\frac{\sum_{i=1}^{d} \lambda_i^2}{d}\right)^{N-n-3} \cdot \left(\frac{d-k-1}{d-k}\right)^{N-n-k-2} \|(A\mathbf{X}[n+1])^{\perp L_{n+1}}\|^2\right]. \quad (67)$$

Now, we can swap the expectations since $\Phi[n+1]$ and $\mathbf{B}[n+k+1]$ are independent.

$$\min_{U[n]} \mathbb{E}_{\mathbf{B}[n+k+1],\Phi[n+1]}[\|(A\mathbf{X}_{n+1})[n+1]^{\perp L_{n+1}}\|^2]$$
$$= \min_{U[n]} \mathbb{E}_{\Phi[n+1]}[\mathbb{E}_{\mathbf{B}[n+k+1]}[\|(A\mathbf{X}_{n+1})[n+1]^{\perp L_{n+1}}\|^2]] \quad (68)$$

As in the previous section, we apply lemma 4.3 to this, since

$$(A\mathbf{X}_{n+1}[n+1])^{\perp L_{n+1}} = ((A\mathbf{X}_{n+1}[n+1])^{\perp \mathcal{T}_{n+1}})^{\perp (\mathbf{B}_{n+1}[n+k+1]^{\perp \mathcal{T}_{n+1}})}. \quad (69)$$

$(A\mathbf{X}_{n+1}[n+1])^{\perp \mathcal{T}_{n+1}}$ is a $d-k$ dimensional vector, and $\mathbf{B}_{n+1}[n+k+1]^{\perp \mathcal{T}_{n+1}}$ is a 1-dimensional vector chosen uniformly at random in that space, so by Lemma 3.1,

$$\mathbb{E}_{\mathbf{B}[n+k+1]}[\|((A\mathbf{X}_{n+1}[n+1])^{\perp \mathcal{T}_{n+1}})^{\perp (\mathbf{B}_{n+1}[n+k+1]^{\perp \mathcal{T}_{n+1}})}\|^2]$$
$$= \left(\frac{d-k-1}{d-k}\right) \|(A\mathbf{X}_{n+1}[n+1])^{\perp \mathcal{T}_{n+1}}\|^2. \quad (70)$$

From eq. (68) we are left with

$$\min_{U[n]} \mathbb{E}_{\Phi[n+1]}[\|(A\mathbf{X}[n+1])^{\perp \mathcal{T}_{n+1}}\|^2] \quad (71)$$
$$= \min_{U[n]} \mathbb{E}_{\Phi[n+1]}[\|A\Phi[n+1] (\Phi[n+1]^{-1}A^{-1}((A\mathbf{X}[n+1])^{\perp \mathcal{T}_{n+1}}))\|^2]$$
$$= \left(\frac{\sum_{i=1}^{d} \lambda_i^2}{d}\right) \min_{U[n]}\{ \mathbb{E}_{\Phi[n+1]}[\|\Phi[n+1]^{-1}A^{-1}((A\mathbf{X}[n+1])^{\perp \mathcal{T}_{n+1}})\|^2]\},$$

using Lemma 5.1. (Note that the vector being scaled is not random because of the lookahead.)

Now consider the projection $(A\mathbf{X}[n+1])^{\perp \mathcal{T}_{n+1}}$, and let $v_{n+1}^*[n+1], \ldots, v_{n+1}^*[n+k]$ be the unique values that achieve this projection, i.e.,

$$(A\mathbf{X}[n+1])^{\perp \mathcal{T}_{n+1}} = A\mathbf{X}[n+1] + v_{n+1}^*[n+1]\mathbf{B}_{n+1}[n+1] + \cdots + v_{n+1}^*[n+k]\mathbf{B}_{n+1}[n+k] \quad (72)$$

Then, we can multiply by $\Phi[n+1]^{-1}A^{-1}$, and change the frame-of-reference for the lookahead $\mathbf{B}$ vectors. We substitute $\mathbf{X}[n+1] = A\mathbf{X}[n] + U[n]\mathbf{B}_n[n]$.

$$\Phi[n+1]^{-1}A^{-1}((A\mathbf{X}[n+1])^{\perp \mathcal{T}_{n+1}}) = A\mathbf{X}[n] + U[n]\mathbf{B}_n[n] + v_{n+1}^*[n+1]\mathbf{B}_n[n+1] + \cdots + v_{n+1}^*[n+k]\mathbf{B}_n[n+k]. \quad (73)$$

Now consider another projection,

$$(A\mathbf{X}[n])^{\perp L_n} = \min_{U_n[n],\ldots,U_n[n+k]} (A\mathbf{X}[n] + U_n[n]\mathbf{B}_n[n] + U_n[n+1]\mathbf{B}_n[n+1] + \cdots + U_n[n+k]\mathbf{B}_n[n+k]). \quad (74)$$

Let the optimal values above be $u_n^*[n], \ldots, u_n^*[n+k]$. Then, for any $U[n]$ the optimum values give a lower bound for the expression in eq. (73),

$$A\mathbf{X}[n] + u_n^*[n]B_n[n] + \cdots + u_n^*[n+k]\mathbf{B}_n[n+k]$$
$$\leq A\mathbf{X}[n] + U[n]B_n[n] + v_{n+1}^*[n+1]\mathbf{B}_n[n+1] + \cdots + v_{n+1}^*[n+k]\mathbf{B}_n[n+k]. \quad (75)$$

Hence, the optimizing values for eq. (68) must be $U[n] = u_n^*[n]$, $v_{n+1}^*[n+1] = u_n^*[n+1], \ldots, v_{n+1}^*[n+k] = u_n^*[n+k]$. And using eq. (73)

$$\min_{U[n]} \mathbb{E}_{\Phi[n+1]}[\|\Phi[n+1]^{-1}A^{-1}((A\mathbf{X}[n+1])^{\perp \mathcal{T}_{n+1}})\|^2] \quad (76)$$
$$= \min_{U[n]} \mathbb{E}_{\Phi[n+1]}[A\mathbf{X}[n] + U[n]\mathbf{B}_n[n] + \cdots + v_{n+1}^*[n+k]\mathbf{B}_n[n+k].]$$
$$= \min_{U[n]}[A\mathbf{X}[n] + U[n]\mathbf{B}_n[n] + \cdots + v_{n+1}^*[n+k]\mathbf{B}_n[n+k].]$$
$$= \|(A\mathbf{X}[n])^{\perp \mathcal{L}_n}\|^2, \quad (77)$$

by eq. (75). The expectation over $\Phi[n+1]$ drops out because there is effectively no randomness due to the lookahead. Therefore,

$$J_n^*(A\mathbf{X}[n], \mathbb{S}_n) = \left(\frac{\sum_{i=1}^{d} \lambda_i^2}{d}\right)^{N-n-2} \left(\frac{d-k-1}{d-k}\right)^{N-n-k-1} \|(A\mathbf{X}[n])^{\perp L_n}\|^2$$

■

Now to prove the main Thm. 5.2.

Proof: We can apply the Lemma to get

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} = \frac{\mathbb{E}_{\mathbb{S}_0,\Phi[0]}[J_0^*(\mathbf{x}[0], \mathbb{S}_0)]}{\|\mathbf{X}[0]\|^2} \quad (78)$$
$$= \left(\frac{\sum_{i=1}^{d} \lambda_i^2}{d}\right)^{N-2} \left(\frac{d-k-1}{d-k}\right)^{N-k-1} \frac{\mathbb{E}_{\mathbb{S}_0,\Phi[0]}[\|(A\mathbf{X}[0])^{\perp L_0}\|^2]}{\|\mathbf{x}[0]\|^2}$$

By Lemma 3.1,

$$\mathbb{E}_{\mathbb{S}_0,\Phi[0]}[\|(A\mathbf{X}[0])^{\perp L_0}\|^2] = \frac{d-k-1}{d}\mathbb{E}_{\Phi[0]}[\|A\mathbf{X}[0]\|^2]$$

By Lemma 5.1,

$$\mathbb{E}_{\Phi[0]}[\|A\mathbf{X}[0]\|^2] = \frac{\sum_{i=1}^{d} \lambda_i^2}{d}\|\mathbf{x}[0]\|^2 \quad (79)$$

Therefore,

$$\frac{\mathbb{E}[\|\mathbf{X}[N]\|^2]}{\|\mathbf{x}[0]\|^2} = \left(\frac{\sum_{i=1}^{d} \lambda_i^2}{d}\right)^{N-1} \frac{d-k-1}{d} \left(\frac{d-k-1}{d-k}\right)^{N-k-1}$$

■

VI. Conclusion

In this paper, we used a simple model of a single uniformly-at-random control direction to capture a sense of unpredictability in actuation. This creates a simpler model of actuation uncertainty than the prior "dropped control" models in [3], [4] because of three reasons:

- This allows us to have trivial identity-matrix dynamics for the plant and still have scalar controls. In the traditional dropped control models, this would not be interesting because the system would not be controllable. The random actuation directions allow our system model to be controllable.
- In the dropped control model, the unpredictability that comes from the dropped control has two effects: it stops us from planning perfectly – since we don't know which future controls will be dropped – and it also causes the system to grow without us being able to act. In our basic model, there is just the unpredictability dimension. If we want to add system growth or dynamics, we can do that in a more orthogonal way.
- In the dropped control model, it is harder to modulate the amount of actuation uncertainty. We can give preview information about the future drops as in [14], [15], but this is not quite satisfying because of the dual effect of a control drop. In the model here we can just give lookahead information to reduce the actuation uncertainty.

Within this model, we found something intriguing: that the second-moment control capacity seems to have a form that parallels the capacity of a bandlimited AWGN channel. The ambient dimension seems to play the role of the bandwidth while the role of SNR involves the amount of lookahead. The results help bridge the world of dimensions (natural to the problem) with the world of bits (growth-rates).

There are many directions in which this model can be taken forward. First, the results here should be extended to the case of having multiple simultaneous control directions to work with. This would involve extending to randomly chosen subspaces instead of simply points on the unit sphere. Second, the interaction with nontrivial system dynamics needs to be explored through considering a sequence of steadily more interesting A matrices. Third, the results need to be extended beyond the second-moment sense of control capacity. One of the key insights from [15], [6] is that the second-moment sense is just one of many. In particular, for the dropped-control model in the scalar case, although the second-moment control capacity is finite, the "Shannon sense" control capacity is infinite. It is our belief that for the model given in this paper, all senses of control capacity are finite.

VII. Appendix

A. Proof of Lemma 4.3

Since $\mathbf{x}^{\|G} \in G$,

$$\mathbf{x}^{\|G} = (\mathbf{x}^{\|G})^{\|G_1} + (\mathbf{x}^{\|G})^{\|G_2} \tag{80}$$

Since $G_1, G_2 \subset G$, $\mathbf{x}^{\|G} = \mathbf{x}^{\|G_1} + \mathbf{x}^{\|G_2}$. Since $G_2 \subset G_1^{\perp}$, $\mathbf{x}^{\|G_2} = (\mathbf{x}^{\perp G_1})^{\|G_2}$. Therefore

$$\mathbf{x}^{\|G} = \mathbf{x}^{\|G_1} + (\mathbf{x}^{\perp G_1})^{\|G_2} \tag{81}$$

$$\mathbf{x} - \mathbf{x}^{\|G} = \mathbf{x} - \mathbf{x}^{\|G_1} - (\mathbf{x}^{\perp G_1})^{\|G_2} \tag{82}$$

$$\mathbf{x}^{\perp G} = \mathbf{x}^{\perp G_1} - (\mathbf{x}^{\perp G_1})^{\|G_2} \tag{83}$$

$$= (\mathbf{x}^{\perp G_1})^{\perp G_2} \tag{84}$$

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