Rational Integration, Simplified

Richard Fateman

June 12, 2009

Abstract

After all this computer algebra system stuff, and several PhD theses in the last few decades, what more could we say about symbolic rational function integration?

How about a closed formula for the result, subject to a few algebraic side-conditions, which works even with parameters in the denominator?

1 Cubic Denominator Example

We start with a simple example: a "proper" rational function integrand with a cubic denominator.

Without loss of generality we set two leading coefficients to unity because we can divide out any higher terms in the numerator and we can move a rational factor outside the integral. The problem looks like this:

$$Q = \frac{x^2 + b\,x + c}{x^3 + d\,x^2 + e\,x + f}$$

The answer to the indefinite integral of Q wrt x can be computed by any number of computer algebra systems. It is huge. Here's a smaller version.

Our only loss of generality is in this assumption: the three roots of the denominator are distinct. Call them $\{r_1,\ r_2,\ r_3\}$ They obey these equations:

$$d + r_3 + r_2 + r_1 = 0$$

$$e + (-r_2 - r_1)\, r_3 - r_1\, r_2 = 0$$

$$f + r_1\, r_2\, r_3 = 0$$

What, now, is the integral of Q?

$$\frac{\left(c + r_3\, b + r_3^2\right)\, \log\left(x - r_3\right)}{\left(r_3 - r_1\right)\, \left(r_3 - r_2\right)} - \frac{\left(c + r_2\, b + r_2^2\right)\, \log\left(x - r_2\right)}{\left(r_2 - r_1\right)\, \left(r_3 - r_2\right)} + \frac{\left(c + r_1\, b + r_1^2\right)\, \log\left(x - r_1\right)}{\left(r_2 - r_1\right)\, \left(r_3 - r_1\right)}$$

There is a pattern here, and so let us define

$$n_i := r_i^2 + b\, r_i + c$$

and also

$$d_{i,j} := r_i - r_j$$

The integration answer is now

$$\frac{n_3\, \log\left(x - r_3\right)}{d_{3,1}\, d_{3,2}} - \frac{n_2\, \log\left(x - r_2\right)}{d_{2,1}\, d_{3,2}} + \frac{n_1\, \log\left(x - r_1\right)}{d_{2,1}\, d_{3,1}}$$

2 Generalizing

The generalization to higher degree is not difficult. For example, we let n_i always be the numerator with r_i substituted for x. If the denominator is degree 4, there are 4 terms in the answer, each looking something like this term (ignoring the sign):

$$\frac{n_4 \log(x - r_4)}{d_{4,1}\, d_{4,2}\, d_{4,3}}$$

The equations relating the (distinct) roots to the coefficients in the denominator naturally involves the additional coefficient. These equations are related to the Newton-Girard formulas, but in any case they can be conveniently computed by equating powers of x in

$$\prod_{i=1}^{n} x - r_i$$

and the denominator of the integrand.

3 As opposed to

In case it occurs to you that this is all too complicated, recall that we can indeed explicitly compute the answer to the original problem using the cubic formula, and substitute it in the integral. Our typesetting expertise was insufficient to stuff the answer on this page, and so we tried for something smaller. *Each* of the 21 occurrences of r_i will then look something like this:

$$\left(-\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right) \sqrt[3]{\frac{\sqrt{27\,f^2 + (4\,d^3 - 18\,d\,e)\;f + 4\,e^3 - d^2\,e^2}}{6\,\sqrt{3}} - \frac{27\,f - 9\,d\,e + 2\,d^3}{54}}$$
$$+ \frac{\left(\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)\left(d^2 - 3\,e\right)}{9\,\sqrt[3]{\frac{\sqrt{27\,f^2+(4\,d^3-18\,d\,e)\;f+4\,e^3-d^2\,e^2}}{6\,\sqrt{3}} - \frac{27\,f-9\,d\,e+2\,d^3}{54}}} - \frac{d}{3}.$$

While the differences in $d_{i,j}$ might simplify somewhat, it is not a happy prospect. Even if we use the “Rootsum” construction, just one term with seven occurrences of this formula is a mess.

4 Writing out the program

We leave as an exercise for the reader, using any convenient computer algebra system, the generation of the integration result for a rational expression such as Q, given the degree, the two lists of coefficients in ascending order in the numerator and the denominator, and the variable of integration (here, x).

5 Limitations

We remind the reader that we are assuming the denominator of the integral has no repeated roots. If there are two equal roots, then division by zero will be implied by part of the formula. If you know there are repeated roots, you have a (cheap) factorization, and hence a partial fraction expansion. This should be applied before reaching this stage.

6 Isn't this well known?

The expressions for the integral in terms of the roots are hardly new: it is a result of residue theory. The partial fraction expansion of a rational function R can be computed from the distinct roots of the denominator: the coefficient of $1/(x - r_j)$ is the residue of R at r_j. This leads directly to the integration formula by integrating $C/(x - r_j)$ to $C \log(x - r_j)$. The point is that by deciding beforehand that you are going to express the result in terms of the roots, you re-represent the answer in a much smaller form. In appropriate circumstances, including perhaps more capable computer algebra programs, this form will be relatively easier to manipulate further.

We should also point out that another expression defining the roots could be used, even smaller, and this is used in the Rootsum expression. Is this better than the technique here? It's hard to say for sure what will work better without some context.

7 Thanks

Thanks to Robert Israel for clarifying comments.