An Example of Analysis Finessing a CAS

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I was asked recently about how the world (or specifically, Matlab) got around the obvious need for a symbolic computer algebra system (CAS) for so long. The answer is, with a little cleverness, you can sometimes do better.

Here is a problem (from a well-regarded but dated 1968 book, *Computing Methods for Scientists and Engineers*, by L. Fox and D.F. Mayers):

Consider the evaluation for successive integer values $r = 1,\ 2,\ \cdots$ of the definite integral

$$I_r \ = \ \int_0^1 e^{4/3(x-1)} x^{r+3} dx.$$

There are a number of possibilities to pursue, one of which is to ask a CAS to express the answer in closed form. This result uses the Incomplete Gamma function.

$$\frac{e^{-\frac{4}{3}}\, 4^{-4-r}\, 3^{4+r}\, \left(\Gamma\left(r+4,0\right) - \Gamma\left(r+4,-\frac{4}{3}\right)\right)}{\left(-1\right)^r}$$

Or we could observe (along with Fox and Mayers) that with a single integration by parts, we can produce the two-term recurrence

$$I_r \ = \ 3/4 - 3/4(r+3)I_{r-1}.$$

I don't know of particular tool in any CAS that facilitates this observation, but you can chose $u,\ dv$ as $u = x^{r+3},\ \ dv = \exp(4/3(x-1))dx$ and proceed with a pencil and paper.

Next, observe that $I_r \to 0$ as $r \to \infty$ which means that we can solve this using numerical means by letting (say) $I_{100} = 0$ and computing other values by a *downward* recurrence:

$$I_r = (3 - 4I_{r+1})/(3r + 12).$$

This form is quite accurate, especially when compared to the CAS closed form above, which, as r grows even modestly, depends on a small difference of two large terms.

Admittedly there was a piece of guesswork here: was 100 large enough? I tried 50 and got the same value for I_2, to full double-float accuracy. I was convinced. Checking... I_{12} computed from the closed-form formula in 40 digit arithmetic:
0.05793726321672580460763426936079208882944 compares to
0.0579372*90750789* in double-float, incorrect digits in italics, vs.
0.0579372632167258 in double-float from recurrence, (all digits correct).

Moral of the story: Closed-form formulae may not be the computational best choice.