Dots, Spaces

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Basic set theory includes notations for "for some x, the predicate $P(x)$ is true" as $\exists x.P(x)$ What does the dot mean there? It is a kind of separator between the "binding of x" and the predicate that follows. We also see this in the lambda-calculus notation $\lambda x \ . \ e$.

A version of the lambda-calculus with types can be confusingly typeset as in Winskel's text [3], through insertion of some spaces. This notation is: $\lambda x \ \in \ X.e$. This lambda expression is intended to denote a functional relationship that can be thought of as the set of all pairs (x, e) where x is the input and e is presumably an expression denoting a computation involving that particular x. The relation holds for members x of the set X. I would have typeset it as $\lambda(x \in X) \ . \ e$ We've used the pair notation (x, e) above because that is what is used in the introduction. Pairs are enclosed in angle brackets subsequently, although it may be that the brackets are used to denote computation and state pairs. That is, we see $\langle c, \sigma \rangle$.

We then see rules such as the one below, defined via this notation:

$$\frac{\langle a_0, \sigma\rangle \rightarrow n_0 \quad \langle a_1, \sigma\rangle \rightarrow n_1}{\langle a_0 + a_1, \sigma\rangle \rightarrow n}$$

where n is the sum of n_0 and n_1.

Here the meaning is "If we are given the premises $\langle a_0, \sigma\rangle \rightarrow n_0$ and $\langle a_1, \sigma\rangle \rightarrow a_1$ then we conclude $\langle a_0 + a_1, \sigma\rangle \rightarrow n$.

Note that the logical connective *and* is denoted by a space. Also note that the divide bar is used for "then" and that the association of the semantics of the summation is adjoined in ordinary English.

The point? Even in a book devoted to Formal Semantics of Programming Languages, informality in notation, to the point of ignoring precedence, overloading symbols and spaces, and providing potential ambiguity is common and apparently understood.

Oh well.

Sometimes a space means "DIVIDE". The "/" has an interesting ambiguity. a/bc written as `$a / b c$` which might be transcribed in TEX as `${a \over b c}$` is displayed as $\frac{a}{bc}$. This is different from the usual programming language interpretation in which we would first substitute a multiplication operator for the space between b and c making it `a / b * c` and then providing the computational semantics of $\frac{a}{b} \cdot c$. The argument for the TEX version is that no human mathematicican would use $1/2\pi$ to mean $\pi/2$. Therefore the former expression must mean $1/(2\pi)$. Does this happen? Sure. Here's one from Courant and Hilbert vol 1 [1] where p. 349 has $u_n = \sqrt{2/\pi\rho_0} \sin nx$ by which is meant

$$u_n = \sqrt{\frac{2}{\pi \cdot \rho_0}} \sin(n \cdot x).$$

Perhaps because of the opportunity for misunderstandings, the occurrences of "/" in typeset mathematics are not nearly as common as they are in computer programs. It seems that for older printed work or texts, the horizontal divide bar is preferred. Slashes are used when space is tight (common in journals today), or a division must be denoted in an exponent, or the division is in a smaller font or is being typeset inline, or perhaps the division occurs in a formula's side-conditions. A typical example of a use in display form:

$$\cdots \left(\frac{x-y}{z}\right)^{v/2} \cdots$$

in [2] p. 619 3.2.4.2. In fact in this large table, the denominators following the "/" are almost always single digit numbers, with rare occurrences of single symbols e.g. $1/r$. When the denominator is larger there might be parentheses: e.g. on p 337, $b^{(1-2\alpha)/(2r)}$. An alternative notation that is also used: $a(b+c)^{-1}$ makes the slash entirely dispensable when the denominator is more than a few characters.

Of course slashes are used in dy/dx or even d/dx as operators, additionally complicating the lives of simple-minded parsers.

Though TEX doesn't seem to have shallow slashes, these are sometimes used for small rational fractions. Thus instead of $1/2$ we have something like ½ where the numerator 1 is shrunk and slid up a bit and the 2 is similarly moved down a bit. While $1/2\pi$ may be ambiguous, ${}^1\!/_2\pi$ is $\pi/2$. Unless of course it is another way of writing $2^{-\pi}$.

References

[1] Courant and Hilbert Methods of Mathematical Physics, volume 1. Wiley, (1953).

[2] A. Prudnikov, P. Brichkov, O. Marichev. Integrals and Series (Moscow) 1983.

[3] Glynn Winskel, *Formal Semantics of Programming Languages*, MIT Press.