Homework Assignment #1

Due date: 9/11/14, in class.

Exercise 1 (Convexity of functions)

1. For $x, y$ both positive scalars, show that

$$ ye^{x/y} = \max_{\alpha > 0} \alpha (x + y) - y \alpha \log \alpha. $$

Use the above result to prove that the function $f$ defined as

$$ f(x, y) = \begin{cases} 
 ye^{x/y} & \text{if } x > 0, \ y > 0, \\
 +\infty & \text{otherwise},
\end{cases} $$

is convex.

2. Show that for $r \geq 1$, the function $f_r : \mathbb{R}^m_+ \to \mathbb{R}$, with values

$$ f_r(v) = \left( \sum_{j=1}^{m} v_j^{1/r} \right)^r $$

is concave.

Exercise 2 (Deleting a measurement in least-squares) We are given a full column rank matrix $A \in \mathbb{R}^{m,n}$, with rows $a_i^\top$, $i = 1, \ldots, m$, and a vector $y \in \mathbb{R}^m$, and a solution to the least-squares problem

$$ x^* = \arg \min_x \sum_{i=1}^{m} (a_i^\top x - y_i)^2 = \arg \min_x \|Ax - y\|_2. $$

Assume now we delete the last measurement, that is, replace $(a_m, y_m)$ by $(0, 0)$. We assume that the matrix obtained after deleting any one of the measurements is still full column rank.

1. Express the solution to the problem after deletion, in terms of the original solution $x^*$. Make sure to explain why any quantities you invert are positive.
2. In the so-called leave-one-out analysis, we would like to efficiently compute all the $m$ solutions corresponding to deleting one of the $m$ measurements. Explain how you would compute those solutions computationally efficiently. Detail the number of operations (flops) needed. You may use the fact that to invert a $n \times n$ matrix costs $O(n^3)$.

**Exercise 3 (Magnitude least-squares)** For given $n$-vectors $a_1, \ldots, a_m$, we consider the problem

$$p^* = \min_x \sum_{i=1}^m (|a_i^\top x| - 1)^2.$$


2. Show that the optimal value $p^*$ depends only on the matrix $K = A^\top A$, where $A = [a_1, \ldots, a_m]$ is the $n \times m$ matrix of data points (that is, if two different matrices $A_1, A_2$ satisfy $A_1^\top A_1 = A_2^\top A_2$, then the corresponding optimal values are the same).

**Exercise 4 (A trust-region problem)** A version of the so-called (convex) trust-region problem amounts to finding the minimum of a convex quadratic function over an Euclidean ball, that is

$$\min_x \frac{1}{2} x^\top H x + c^\top x + d$$

s.t. $x^\top x \leq r^2,$

where $H \succ 0$, and $r > 0$ is the given radius of the ball. Prove that the optimal solution to this problem is unique and it is given by

$$x(\lambda^*) = -(H + \lambda^* I)^{-1} c,$$

where $\lambda^* = 0$ if $\|H^{-1} c\|_2 \leq r$, or otherwise $\lambda^*$ is the unique value such that

$$\|(H + \lambda^* I)^{-1} c\|_2 = r.$$

**Exercise 5 (Sum of top eigenvalues)** For $X \in \mathbb{S}^n$, and $i \in \{1, \ldots, n\}$, we denote by $\lambda_i(X)$ the $i$-th largest eigenvalue of $X$. For $k \in \{1, \ldots, n\}$, we define the function $f_k : \mathbb{S}^n \to \mathbb{R}$ with values

$$f_k(X) = \sum_{i=1}^k \lambda_i(X).$$

This function is an intermediate between the largest eigenvalue (obtained with $k = 1$) and the trace (obtained with $k = n$).
1. Show that for every $t \in \mathbb{R}$, we have $f_k(X) \leq t$ if and only if there exist $Z \in S^n$ and $s \in \mathbb{R}$ such that

$$t - ks - \text{trace}(Z) \geq 0, \quad Z \succeq 0, \quad Z - X + sI \succeq 0.$$ 

2. Show that $f_k$ is convex. Is it a norm?

3. How would you generalize these results to the function that assigns the sum of top $k$ singular values to a general rectangular $m \times n$ matrix, with $k \leq \min(m, n)$?