Quiz 3

NAME: 
SID: 

The quiz lasts one hour. Notes are allowed.

1. Consider the optimization problem

\[ p^* := \min_x f(x) := \sum_{i=1}^n y_i^2 h(x_i) : \sum_{i=1}^n x_i = 1. \]

where \( y \in \mathbb{R}^n \) is given, with \( y_i \neq 0 \) for every \( i = 1, \ldots, n \). In the above, \( h : \mathbb{R} \to \mathbb{R} \) is the function

\[ h(\xi) = \begin{cases} 
\frac{1}{\xi} & \text{if } \xi > 0, \\
+\infty & \text{otherwise}.
\end{cases} \]

(a) Is the problem convex?
(b) Show that the dual function can be written as

\[ g(\nu) = \begin{cases} 
2\|y\|_1 \sqrt{\nu} - \nu & \text{if } \nu \geq 0, \\
-\infty & \text{otherwise}.
\end{cases} \]

(c) Solve the dual problem and find its optimal value \( d^* \).
(d) Use the dual to guess a feasible point \( x^* \) for the primal problem, such that \( f(x^*) = d^* \).
(e) Show that strong duality holds, and find an optimizer for the primal problem.
2. Let $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$ and $\mu > 0$. Consider the problem

$$\min_x \|Ax - y\|_1 + \mu \|x\|_2.$$ 

(a) Express the problem in standard SOCP format.

(b) Find a dual to the problem. *Hint:* use the fact that, for any vector $z$:

$$\max_{u : \|u\|_2 \leq 1} u^T z = \|z\|_2, \quad \max_{u : \|u\|_\infty \leq 1} u^T z = \|z\|_1.$$

(c) Does strong duality hold? *Hint:* look at the dual problem.

(d) Assume $A$ is $100 \times 10^6$. Which problem would you solve, the primal or the dual? Justify your answer carefully.