Lecture 10: **SOCP Duality**

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Reading assignment: Chapter 5 of BV.

10.1 Duality in second-order cone optimization

Second-order cone optimization is a special case of semi-definite optimization. It is, however, instructive to develop a more direct approach to duality for SOCPs.

10.1.1 Conic approach

We start from the second-order cone problem in inequality form:

$$\min_x \; c^T x \; : \; \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \;\; i = 1, \ldots, m,$$

where $c \in \mathbf{R}^n$, $A_i \in \mathbf{R}^{n_i \times n}$, $b_i \in \mathbf{R}^{n_i}$, $c_i \in \mathbf{R}^n$, $d_i \in \mathbf{R}$, $i = 1, \ldots, m$.

Conic approach. To build a Lagrangian for this problem, we use the fact that, for any pair (t, y):

$$\max_{(u,\lambda)\,:\,\|u\|_2 \leq \lambda} u^T y - t\lambda = \max_{\lambda \geq 0} \lambda(\|y\|_2 - t) = \begin{cases} 0 & \text{if } \|y\|_2 \leq t \\ +\infty & \text{otherwise.} \end{cases}$$

Geometrically, the second-order cone has a 90^o angle at the origin.

Consider the following Lagrangian, with variables x, $\lambda \in \mathbf{R}^m$, $u_i \in \mathbf{R}^{n_i}$, $i = 1, \ldots, m$:

$$\mathcal{L}(x, \lambda, u_1, \ldots, u_m) = c^T x + \sum_{i=1}^m \left(u_i^T (A_i x + b_i) - \lambda_i (c_i^T x + d_i) \right).$$

Using the fact above leads to the following minimax representation of the primal problem:

$$p^* = \min_x \max_{\|u_i\|_2 \leq \lambda_i,\, i=1,\ldots,m} \mathcal{L}(x, \lambda, u_1, \ldots, u_m).$$

Weak duality expresses as $p^* \geq d^*$, where

$$d^* = \max_{\lambda, u_i,\, i=1,\ldots,m} \min_x \mathcal{L}(x, \lambda, u_1, \ldots, u_m).$$

The inner problem, which corresponds to the dual function, is very easy to solve as the problem is unconstrained and the objective affine (in x). Setting the derivative with respect to x leads to the dual constraints

$$\sum_{i=1}^{m} A_i^T u_i - \lambda_i c_i = 0.$$

We obtain

$$d^* = \max_{\lambda, u_i,\, i=1,\ldots,m} \lambda^T d + \sum_{i=1}^{m} u_i^T b_i \;:\; \sum_{i=1}^{m} A_i^T u_i - \lambda_i c_i = 0, \;\; \|u_i\|_2 \le \lambda_i, \;\; i = 1, \ldots, m.$$

The above is an SOCP, just like the original one.

10.1.2 Direct approach

As for the SDP case, it turns out that the above "conic" approach is the same as if we had used the Lagrangian

$$\mathcal{L}_{\text{direct}}(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i \left(\|A_i x + b_i\|_2 - (c_i^T x + d_i) \right).$$

Indeed, we observe that

$$\mathcal{L}_{\text{direct}}(x, \lambda) = \max_{u_i,\, i=1,\ldots,m} \mathcal{L}(x, \lambda, u_1, \ldots, u_m) \;:\; \|u_i\|_2 \le v_i, \;\; i = 1, \ldots, m.$$

10.1.3 Strong duality

Strong duality results are similar to those for SDP: a sufficient condition for strong duality to hold is that one of the primal or dual problems is strictly feasible. If both are, then the optimal value of both problems is attained.

10.1.4 Examples

Minimum distance to an affine subspace. Return to the problem seen in lecture 11:

$$p^* = \min \|x\|_2 \;:\; Ax = b, \tag{10.1}$$

where $A \in \mathbf{R}^{p \times n}$, $b \in \mathbf{R}^p$, with b in the range of A. We have seen how to develop a dual when the objective is squared. Here we will work directly with the Euclidean norm.

The above problem is an SOCP. To see this, simply put the problem in epigraph form. Hence the above theory applies. A more direct (equivalent) way, which covers cases when norms appear in the objective, is to use the representation of the objective as a maximum:

$$p^* = \min_x \max_{\nu,\, \|u\|_2 \le 1} x^T u + \nu^T (b - Ax) \ge d^* = \max_{\nu,\, \|u\|_2 \le 1} \min_x x^T u + \nu^T (b - Ax).$$

The dual function is

$$g(u) = \min_x \; x^T u + \nu^T(b - Ax) = \begin{cases} \nu^T b & \text{if } A^T\nu = u, \\ -\infty & \text{otherwise.} \end{cases}$$

We obtain the dual

$$d^* = \max_{\nu,\, u} \; b^T\nu \;:\; A^T\nu = u, \;\; \|u\|_2 \le 1.$$

Eliminating u:

$$d^* = \max_{\nu} \; b^T\nu \;:\; \|A^T\nu\|_2 \le 1.$$

Robust least-squares. Consider a least-squares problem

$$\min_x \; \|Ax - b\|_2,$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$. In practice, A may be noisy. To handle this, we assume that A is additively perturbed by a matrix bounded in largest singular value norm (denoted $\|\cdot\|$ in the sequel) by a given number $\rho \ge 0$. The robust counterpart to the least-squares problem then reads

$$\min_x \; \max_{\|\Delta\| \le \rho} \; \|(A + \Delta)x - b\|_2.$$

Using convexity of the norm, we have

$$\forall\, \Delta, \;\; \|\Delta\| \le \rho \;:\; \|(A+\Delta)x - b\|_2 \le \|Ax - b\|_2 + \|\Delta x\|_2 \le \|Ax - b\|_2 + \rho\|x\|_2.$$

The upper bound is attained, with the choice[1]

$$\Delta = \frac{\rho}{\|x\|_2 \cdot \|Ax - b\|_2}(Ax - b)x^T.$$

Hence, the robust counterpart is equivalent to the SOCP

$$\min_x \; \|Ax - b\|_2 + \rho\|x\|_2.$$

Again, we can use epigraph representations for each norm in the objective:

$$\min_{x,t,\tau} \; t + \rho\tau \;:\; t \ge \|Ax - b\|_2, \;\; \tau \ge \|x\|_2.$$

and apply the standard theory for SOCP developed in section 10.1.1. Strong duality holds, since the problem is strictly feasible.

An equivalent, more direct approach is to represent each norm as a maximum:

$$p^* = \min_x \; \max_{\|u\|_2 \le 1,\, \|v\|_2 \le \rho} \; u^T(b - Ax) + v^T x.$$

[1] We assume that $x \ne 0$, $Ax \ne b$. These cases are easily analyzed and do not modify the result.

Exchanging the min and the max leads to the dual

$$p^* \geq d^* = \max_{\|u\|_2 \leq 1,\, \|v\|_2 \leq \rho} \min_x \; u^T(b - Ax) + v^T x.$$

The dual function is

$$g(u, v) = \min_x \; v^T(b - Ax) + u^T x = \begin{cases} v^T b & \text{if } A^T v + u = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Eliminating u, we obtain the dual

$$d^* = \max_{u,v} \; v^T b \;:\; \|A^T v\|_2 \leq 1, \;\; \|v\|_2 \leq \rho.$$

As expected, when ρ grows, the dual solution tends to the least-norm solution to the system $Ax = b$. It turns out that the above approach leads to a dual that is equivalent to the SOCP dual, and that strong duality holds.

Exercises