EE 227A: Convex Optimization

Lecture 2: Convex Optimization

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Outline

Convex problems
  Convex sets
  Convex functions
  Convex problems

Duality
  Weak duality
  Examples
  Strong duality

References
Convex sets

Definition

A set $C$ in $\mathbb{R}^n$ is convex if, or any two pair of points, such as $x_1$ and $x_2$, the line segment joining the two points is entirely in the set. Mathematically:

$$\forall x_1, x_2 \in C, \ \forall \lambda \in [0, 1] : \lambda x_1 + (1 - \lambda x_2) \in C.$$ 

Points $x_1, x_2$ are in the set, so the line between them is.
Convex sets

Intersection rule

The intersection of convex sets is convex: if $C_\alpha$ is a family of convex sets indexed by $\alpha \in \mathcal{A}$, then $\bigcap_{\alpha \in \mathcal{A}} C_\alpha$ is convex. (We allow for infinite sets $\mathcal{A}$.)

Examples:

- The second-order cone $\{(y, t) : \|y\|_2^2 \leq t\}$ is convex, as it is the intersection of half-spaces of the form $u^T y \leq t$ with $u$ spanning the unit Euclidean sphere.

- The cone of positive, semi-definite (PSD) matrices is convex, since a $n \times n$ symmetric matrix $X$ is PSD iff $z^T X z \geq 0$ for every $n$-vector $z$. 
Convex sets

Affine transformation rule

**Affine transformations** of sets are convex. If $C \subseteq \mathbb{R}^m$ is convex, then for any $\hat{x} \in \mathbb{R}^n$ and $R \in \mathbb{R}^{n \times m}$, the set $\{\hat{x} + Ru : u \in C\}$ is convex.

*Example:* For a given $m \times n$ matrix $A$, $m$-vector $b$, $n$-vector $c$, and scalar $d$, the set

$$\left\{ x \in \mathbb{R}^n : \|Ax + b\|_2 \leq c^T x + d \right\}$$

is convex.
Convex sets
Example: chance constraint

If \( a \in \mathbb{R}^n \) is a Gaussian random variable with mean \( \hat{a} \) and covariance matrix \( \Sigma \), then the set of points \( x \in \mathbb{R}^n \) such that

\[
\text{Prob}\{a : a^T x \leq b\} \geq 0.99
\]

is convex, and can be expressed via the second-order cone condition

\[
\hat{a}^T x + \kappa \sqrt{x^T \Sigma x} \leq b,
\]

\[
= \| \Sigma^{1/2} x \|_2
\]

where \( \kappa = \Phi^{-1}(0.99) \approx 2.33 \), with \( \Phi \) the CDF of the standard Gaussian.
Convex functions
Extended value functions

Extended value functions and domain:

- The **domain** of a function $f$, denoted by $\text{dom} f$, is the set of points where it is finite.
  Example: $f : x \rightarrow \log x$, $\text{dom} f = \mathbb{R}_{++}$.

- We can extend functions outside the domain using the values $\pm \infty$.
  Example: $f : x \rightarrow \log x$, $x > 0$, $-\infty$ otherwise.
Convex functions

Definition

A (extended value) function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if

$$\forall x_1, x_2, \forall \lambda \in [0, 1] : f(\lambda x_1 + (1 - \lambda x_2)) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

(This requires the domain to be convex.)

Examples:

- $f : x \to 1/x$ if $x > 0$, $+\infty$ otherwise, is convex.
- $f : x \to 1/x$ if $x \neq 0$, $+\infty$ otherwise, is **not** convex.
- $f : x \to -\log x$, $x > 0$, $+\infty$ otherwise, is convex.
- $f : x \to -\log x$, $x > 0$, $-\infty$ otherwise, is **not** convex.
Convex functions

Epigraph

Epigraph of a function:

\[ \text{epi } f := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t \}. \]

\( f \text{ convex } \iff \text{epi } f \text{ is convex.} \)
Convex functions

Alternate characterizations

$f$ differentiable, convex $\iff$

$$\forall x, x_0 : f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0).$$

Affine approximation to $f$ at any point is **global** lower bound on $f$. 

![Graph showing affine approximation to a convex function](image-url)
Convex functions

Alternate characterizations

Let $f$ be twice differentiable, convex if and only if its Hessian

$$\nabla^2 f(x) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{1 \leq ij \leq n}$$

is positive semidefinite (PSD) for every $x$.

In general, this condition is hard to check. For quadratic functions though, it is easy, as seen next.
Quadratic functions

Representation via symmetric matrices

A quadratic function \( q : \mathbb{R}^n \to \mathbb{R} \) can be represented as

\[
q(x) = \frac{1}{2} x^T Q x + b^T x + c,
\]

for appropriate \textit{symmetric} matrix \( Q \), vector \( b \in \mathbb{R}^n \), and scalar \( c \).
Positive semidefinite matrices

Flashback from linear algebra:

- A square matrix $A$ has $n$ (possibly non-distinct) eigenvalues, which are (in general complex) numbers that solve $\det(\lambda I - A) = 0$.
- Symmetric matrices have real eigenvalues only.
- A symmetric matrix $Q$ is said to be *positive semi-definite* (PSD) if $\forall x : x^T Q x \geq 0$.

We write $Q \succeq 0$. 
Eigenvalue decomposition for symmetric matrices

Theorem (EVD of symmetric matrices)

*We can decompose any symmetric $p \times p$ matrix $Q$ as*

$$Q = U \Lambda U^T = \sum_{i=1}^{p} \lambda_i u_i u_i^T,$$

*where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$, with $\lambda_1 \geq \ldots \geq \lambda_n$ the eigenvalues, and $U = [u_1, \ldots, u_p]$ is a $p \times p$ orthogonal matrix ($U^T U = I_p$) that contains the eigenvectors of $Q$.*

Corollary

$$Q \succeq 0 \iff \text{every eigenvalue is non-negative}.$$
Convex functions

Convex quadratic functions

A quadratic function \( q : \mathbb{R}^n \rightarrow \mathbb{R} \) can be represented as

\[
q(x) = \frac{1}{2} x^T Q x + b^T x + c,
\]

for appropriate symmetric matrix \( Q \), vector \( b \in \mathbb{R}^n \), and scalar \( c \). Since \( \nabla^2 q(x) = Q \):

Theorem (Convex quadratic functions)

\( q \) convex \iff \( Q \succeq 0 \).
Convex functions

Some properties

- Pointwise maxima of convex functions are convex: if $f_\alpha$ is a family of convex functions sets indexed by $\alpha \in \mathcal{A}$, then $f$ with values $\max_{\alpha \in \mathcal{A}} f_\alpha(x)$ is convex.

- Conversely, any convex function can be represented as maxima of affine ones. (Hence, “convex means max-linear”).

- If $f$ is convex and component-wise monotone increasing and $g_1, \ldots, g_k$ are convex, then $h = f \circ g$ with values $h(x) = f(g_1(x), \ldots, g_k(x))$ is convex.

Proof: the epigraph of $h$ can be represented via convex constraints:

$$h(x) \leq t \iff \exists \ u \in \mathbb{R}^k : f(u) \leq t, \ g_i(x) \leq u_i, \ i = 1, \ldots, k.$$
Convex functions

Examples

The following functions are convex:

- Norms, such as $l_1$, $l_2$ and $l_\infty$. Hence, for example, $x \to \|Ax + b\|_2$, with $A$ a matrix and $b$ a vector, is convex.

- The function $x \to \lambda_{\text{max}}(F(x))$, with $F(x) = F_0 + x_1 F_1 + \ldots + x_m F_m$ and affine combination of given symmetric matrices, and $\lambda_{\text{max}}$ is the largest eigenvalue, is convex.
Convex problems

Definition

The problem in standard form

\[ p^* := \min_x f_0(x) \text{ subject to } f_i(x) \leq 0, \quad i = 1, \ldots, m, \]
\[ Ax = b, \]

is convex if the functions \( f_0, \ldots, f_m \) are all convex. Here \( A \in \mathbb{R}^{p \times n}, \) \( b \in \mathbb{R}^p \) are given.

Note that only affine equality constraints are allowed.
Definition
Maximization problems

The problem in standard form

$$\max \ f_0(x) \ \text{subject to} \ f_i(x) \leq 0, \ i = 1, \ldots, m,$$
$$Ax = b,$$

is convex if

- The function $f_0$ is concave.
- The functions $f_1, \ldots, f_m$ are all convex.
Linear programming (LP) involves the minimization of a linear function over a polytope:

$$\min_{x} c^T_0 x : c^T_{i} x + d_i \geq 0, \quad i = 1, \ldots, m.$$ 

The problem

$$\min_{x} 3x_1 + 1.5x_2$$

s.t. $$-1 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq 3$$

is an LP, since the objective and constraint functions are all affine.
Problem classes

Quadratic programming

Quadratic programming (QP) involves the minimization of a quadratic convex function over a polytope.

$$\min_x c^T x + x^T Q x : c_i^T x + d_i \geq 0, \quad i = 1, \ldots, m.$$ 

Here, $Q$ must be PSD.

The problem

$$\min_x x_1^2 - x_1 x_2 + 2x_2^2$$

$$\text{s.t.} \quad -3x_1 - 1.5x_2$$

$$0 \leq x_2 \leq 3$$

is a QP, since objective is quadratic convex, and the constraint functions are all affine.
Example of a quadratic program

The slalom problem

The problem seen in Overview lecture can be formulated as a QP:

$$\min_{\nu} \nu^T u_0 + \sum_{i=1}^n \frac{(y_i + \nu_{i+1} - \nu_i)^2}{\sigma_i^2} : \|\nu\|_\infty \leq c.$$ 

where $\sigma_i$ (resp. $y_i$) is the horizontal (resp. vertical) distance between the middle of the gates.
**Problem classes**

**Second-order cone programming**

Second-order cone programming (SOCP) generalizes LP and QP via the inclusion of Euclidean norms in the constraint functions.

\[
\min_{x} \quad c_0^T x \quad : \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m.
\]

**Application:** Chance-constrained linear programming

\[
\min_{x} \quad c^T x \quad : \quad \text{Prob}\{a_i^T x \leq b\} \geq 0.99, \quad i = 1, \ldots, m,
\]

where each \(a_i\) is a Gaussian random variable with mean \(\hat{a}_i\) and covariance matrix \(\Sigma_i, i = 1, \ldots, m\).
Semi-definite programming (SDP) involves the minimization of a linear function over the constraint that a symmetric matrix affine in the decision variables be positive-semidefinite:

$$\min_x c_0^T x : F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0.$$  

(Here, $F_0, \ldots, F_n$ are given symmetric matrices.)

*Application*: relaxation of Boolean problems, see later.
Lagrangian

Consider the (non-necessarily convex) “primal” problem in standard form

\[ p^* := \min_x f_0(x) \text{ subject to } f_i(x) \leq 0, \ i = 1, \ldots, m, \]

We can write the problem as an unconstrained one:

\[ p^* := \min_x \max_{y \geq 0} \mathcal{L}(x, y), \]

where \( \mathcal{L} \) is the Lagrange function:

\[ \mathcal{L}(x, y) := f_0(x) + \sum_{i=1}^{m} y_i f_i(x). \]
Min-max inequality

**Theorem (Min-max inequality)**

For any function $\mathcal{L}$ of two variables:

$$\min_x \max_y \mathcal{L}(x, y) \geq \max_y \min_x \mathcal{L}(x, y).$$

**Proof.**

In the LHS, maximizing Player $y$ has full information on $x$ ($y^*$ is a function of $x$). The RHS is better (smaller) for the minimizing Player $x$, as then $y$ has no information on $x$. ■
Weak duality

Weak duality is the process of applying the min-max inequality to the problem in unconstrained form:

\[ p^* = \min_x \max_{y \geq 0} \mathcal{L}(x, y) \geq d^* := \max_{y \geq 0} \min_x \mathcal{L}(x, y). \]

The dual problem is

\[ d^* = \max_{y \geq 0} G(y), \text{ where } G(y) := \min_x \mathcal{L}(x, y) \]

is called the dual function.

Since \( G \) is concave, the dual problem is convex (but not necessarily easy to solve!)
Example

LP duality

Linear programming:

\[ p^* := \min c^T x : Ax \leq b. \]

Lagrangian:

\[ \mathcal{L}(x, y) = c^T x + y^T (Ax - b). \]

Dual is also an LP

\[ p^* \geq d^* = \max \min_{y \geq 0} \mathcal{L}(x, y) = \max_{y \geq 0, A^T y + c = 0} -b^T y. \]

Turns out to be equivalent \((p^* = d^*)\), as seen later.
Example of weak duality
A Boolean problem

Let $W = W^T$ be a given $n \times n$ matrix.

$$p^* := \max_x x^T W x : x_i^2 = 1, \quad i = 1, \ldots, n$$

- Arises in e.g., segmentation problems (with $W$ the Laplacian matrix of a graph).
- Hard combinatorial problem.
- Duality yields a provably good, non-trivial bound.
Example of weak duality

Dual of Boolean problem

We can express the problem in an \textit{unconstrained} way

\[ p^* = \min_x \max_y L(x, y) \]

where \( L \) is the Lagrangian:

\[ L(x, y) := x^T W x + \sum_{i=1}^{n} y_i (1 - x_i^2) \]

Dual function:

\[ G(y) := \max_x L(x, y) = \begin{cases} \sum_{i=1}^{n} y_i & \text{if } \text{diag}(y) - W \text{ is PSD}, \\ +\infty & \text{otherwise}. \end{cases} \]

Dual problem is an SDP:

\[ p^* \leq d^* := \max_y G(y) = \max_y \sum_{i=1}^{n} y_i : \text{diag}(y) \succeq W. \]
Example of weak duality

CVX syntax

CVX is a matlab prototyping tool for convex optimization [1].

Here is a matlab snippet that solves for the dual bound, assuming $W$, $n$ are in the workspace:

```matlab
cvx_begin
    variable y(n,1);
    minimize( sum(y) );
    subject to
        diag(y)-W == semidefinite(n) ;
cvx_end

dstar = sum(y);
```
Strong duality
Definition & main result

Recall weak duality:

\[
p^* = \min_x \max_{y \geq 0} \mathcal{L}(x, y) \geq d^* := \max_{y \geq 0} \min_x \mathcal{L}(x, y).
\]

We say that strong duality holds when \( p^* = d^* \).

- It holds when the primal problem is convex and strictly feasible.
- It holds for LPs or QPs that are feasible.
- It holds for convex problems that are strictly feasible.
- It holds in some very specific non-convex problems. For example, if \( m = 1 \) and both \( f_0, f_1 \) are quadratic.
Example of strong duality
Bidual of Boolean problem

Return to Lagrange relaxation ("dual") of Boolean problem:

\[ d^* = \max_y \sum_{i=1}^n y_i : \text{diag}(y) \succeq W. \]

Express it as max-min problem:

\[ d^* = \max_y \min_{X \succeq 0} \sum_{i=1}^n y_i + \text{Tr} X(\text{diag}(y) - W). \]

Here we use the fact that for any symmetric matrix \( Q \),

\[ \max_{X \succeq 0} \text{Tr} QX = \begin{cases} 0 & \text{if } Q \preceq 0, \\ +\infty & \text{otherwise}. \end{cases} \]

Exchanging min and max leads to the so-called bidual :

\[ p^{**} := \max_X \text{Tr} WX : X \succeq 0, \ X_{ii} = 1, \ i = 1, \ldots, n. \]
Example of strong duality
Bidual of Boolean problem

Bidual:
\[
\max_X \text{Tr } WX : X \succeq 0, \ X_{ii} = 1, \ i = 1, \ldots, n.
\]

- Also an SDP, with objective value equal to \(d^*\) (strong duality holds).
- Can obtain this directly by expressing original problem in terms of matrix variable \(X := xx^T\), and dropping the rank constraint on \(X\).
- Suggests a way to generate primal feasible points from bidual variable \(X\), using distribution with mean 0 and covariance matrix given by \(X\).
- Allows to prove that \(\left(\frac{2}{\pi}\right)d^* \leq p^* \leq d^*\) (independent of problem size!).
Example of strong duality

CVX syntax

Here is a matlab snippet that solves the bidual, assuming $W, n$ are in the workspace:

```matlab
cvx_begin
    variable X(n,n) symmetric;
    maximize( trace(X*W) );
    subject to
        X == semidefinite(n) ;
        diag(X) == 1;

cvx_end
dstar = trace(X*W);
```

Alternatively, we can use the `dual` command in the original dual problem:

```matlab
cvx_begin
    variable y(n,1);
    dual variable X
    minimize( sum(y) );
    subject to
        diag(y)-W == semidefinite(n) : X;

cvx_end
```
Optimality conditions

If the primal problem is convex, and both primal and dual are strictly feasible, then

- Strong duality holds;
- both problems are attained;
- the “Karush-Kuhn-Tucker” (KKT) conditions

\[ x^* \in \arg \min_x \mathcal{L}(x, y^*), \quad y_i^* f_i(x^*) = 0, \quad i = 1, \ldots, m. \]

characterize optimality (primal-dual feasible pairs are optimal iff they satisfy them).
Applications of duality

- Sensitivity analysis of convex problems.
- Stopping criteria in convex optimization algorithms.
- Decomposition methods for large-scale convex optimization.
- Bounds and approximations for non-convex problems.
- For convex problems, leads to often surprising connections.
References

S. Boyd and M. Grant.  
The CVX optimization package, 2010.

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