Quiz 1: Solutions

1. Consider the matrix $A = uv^T$, with $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$.

   (a) Find the nullspace and range of $A$.

   (b) Explain how to compute an SVD of $A$.

Solutions: We assume $u \neq 0$, $v \neq 0$ to avoid trivialities.

(a) For the nullspace, the condition $Ax = 0$ is written $(v^T x)u = 0$. Since $u \neq 0$, we obtain $v^T x = 0$. That is, the nullspace is the hyperplane going through zero of vectors orthogonal to $v$.

For the range, we look at the set of vectors of the form $(v^T x)u$, when $x$ ranges the whole space $\mathbb{R}^n$. Clearly it is included in the line going through the origin with direction $u$, $L := \{tu : t \in \mathbb{R}\}$. Now for any point in $L$, say of the form $tu$ with $t \in \mathbb{R}$, we can find $x$ such that $t = v^T x$; for example $x = (t/v^Tv)v$. Thus the nullspace equals to the line $L$.

(b) We can write $A = \sigma pq^T$, where $p = u/\|u\|_2$, $q = v/\|v\|_2$, $\sigma = \|u\|_2 \cdot \|v\|_2$. This is basically the SVD of $A$ in short form. The full version would require to complete the vector $p$ (resp. $q$) via orthogonalization to form an orthogonal matrix $U$ that contains $p$ as its first column. Similarly we form an orthogonal matrix $V$ that contains $q$ as its first column. Finally, we set $S = \text{diag}(\sigma, 0, \ldots, 0)$ of size $n \times m$. We have

$$USV^T = \sigma pq^T = A.$$ 

This proves that the triple $(U, S, V)$ is an SVD of $A$. 


2. Consider the $2 \times 2$ matrix

$$A = \frac{1}{\sqrt{10}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \frac{2}{\sqrt{10}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}.$$ 

(a) What is an SVD of $A$? Express it as $A = U S V^T$, with $S$ the diagonal matrix of singular values ordered in decreasing fashion. Make sure to check all the properties required for $U, S, V$.

(b) Find the semi-axis lengths and principal axes of the ellipsoid

$$\mathcal{E}(A) = \{Ax : x \in \mathbb{R}^2, \|x\|_2 \leq 1\}.$$ 

*Hint:* Use the SVD of $A$ to show that every element of $\mathcal{E}(A)$ is of the form $y = U \tilde{y}$ for some element $\tilde{y}$ in $\mathcal{E}(S)$. That is, $\mathcal{E}(A) = \{U \tilde{y} : \tilde{y} \in \mathcal{E}(S)\}$. (In other words the matrix $U$ maps $\mathcal{E}(S)$ into the set $\mathcal{E}(A)$.) Then analyze the geometry of the simpler set $\mathcal{E}(S)$.

(c) What is the set $\mathcal{E}(A)$ when we append a zero vector after the last column of $A$, that is $A$ is replaced with $\tilde{A} = [A, 0] \in \mathbb{R}^{2 \times 3}$?

(d) Same question when we append a row after the last row of $A$, that is, $A$ is replaced with $\tilde{A} = [A^T, 0]^T \in \mathbb{R}^{3 \times 2}$. Interpret geometrically your result.

**Solution:**

(a) We have

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = U S V^T,$$

where $U = [u_1, u_2]$, $V = [v_1, v_2]$ and $S = \text{diag}(\sigma_1, \sigma_2)$, with $\sigma_1 = 2$, $\sigma_2 = 1$, and

$$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$ 

The triplet $(U, S, V)$ is an SVD of $A$, since $S$ is diagonal with non-negative elements on the diagonal, and $U, V$ are orthogonal matrices ($U^T U = V^T V = I_2$). To check this, we first check that the Euclidean norm of $u_1, u_2, v_1, v_2$ is one. (This is why we factored the term $\sqrt{10}$ into $\sqrt{2} \cdot \sqrt{5}$.) In addition, $u_1^T u_2 = v_1^T v_2 = 0$. Thus, $U, V$ are orthogonal, as claimed.

(b) We have, for every $x, y := Ax = US(V^T x)$ hence $y = U \tilde{y}$, with $\tilde{y} = S \bar{x}$ and $\bar{x} = V^T x$. Since $V$ is orthogonal, $\|\bar{x}\|_2 = \|x\|_2$. In fact, when $x$ runs the unit Euclidean sphere, so does $\bar{x}$. Thus every element of $\mathcal{E}(A)$ is of the form $y = U \tilde{y}$ for some element $\tilde{y}$ in $\mathcal{E}(S)$. To analyze $\mathcal{E}(A)$ it suffices to analyze $\mathcal{E}(S)$ and then transform the points of the latter set via the mapping $\tilde{y} \rightarrow U \tilde{y}$.

Since

$$\mathcal{E}(S) = \{\sigma_1 \bar{x}_1 e_1 + \sigma_1 \bar{x}_2 e_2 : \bar{x}_1^2 + \bar{x}_2^2 \leq 1\},$$
with $e_1, e_2$ the unit vectors, we have

$$\mathcal{E}(A) = \{ \sigma_1 \bar{x}_1 u_1 + \sigma_1 \bar{x}_2 u_2 : \bar{x}_1^2 + \bar{x}_2^2 \leq 1 \}.$$ 

In the coordinate system defined by the orthonormal basis $(u_1, u_2)$ the set is an ellipsoid with semi-axis lengths $(\sigma_1, \sigma_2)$, and principal axes given by the coordinate axes. In the original system the principal axes are $u_1, u_2$.

(c) When we append a zero column after the last column of $A$ we are doing nothing to $\mathcal{E}(A)$. Indeed, the condition

$$y = Ax \text{ for some } x \in \mathbb{R}^2, \quad \|x\|_2 \leq 1$$

is the same as

$$y = \begin{pmatrix} A & 0 \end{pmatrix} z \text{ for some } z \in \mathbb{R}^3, \quad \|z\|_2 \leq 1.$$ 

Geometrically, the projection of a 3-dimensional unit sphere on the first two coordinates is the 2-dimensional unit sphere. Hence we lose nothing if the 2D sphere used to generate the points $x$ is replaced by the projection of the 3D sphere.

(d) Here we append a row after the last row of $A$, replacing $A$ with

$$\tilde{A} = \begin{pmatrix} A \\ 0 \end{pmatrix} \in \mathbb{R}^{3 \times 2}.$$ 

The set $\mathcal{E}(\tilde{A})$ is the set of points of the form $(y, 0) \in \mathbb{R}^3$ where $y \in \mathcal{E}(A)$. This means that we are simply embedding the ellipsoid $\mathcal{E}(A)$ into a 3D space, instead of the original 2D one. The set $\mathcal{E}(\tilde{A})$ is now a degenerate (flat) ellipsoid in $\mathbb{R}^3$, entirely contained on the plane defined by the first two unit vectors in $\mathbb{R}^3$. 

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