Case Studies in Robust Optimization

Laurent El Ghaoui

Department of EECS, UC Berkeley
A Case Study in Robust Optimization

Laurent El Ghaoui

Department of EECS, UC Berkeley
A Robust Option Pricing Problem

Laurent El Ghaoui

Department of EECS, UC Berkeley
Robust optimization

standard form:

$$\min_x \sup_{u \in \mathcal{U}} f_0(x, u) : \forall u \in \mathcal{U}, \ f_i(x, u) \leq 0, \ i = 1, \ldots, m$$

- $x \in \mathbb{R}^n$ is the decision variable
- $u$ is a parameter vector affecting the problem data
- set $\mathcal{U}$ describes uncertainty on $u$

a semi-infinite optimization problem
Agenda

• option basics
• option pricing problem
• worst-case approach

joint work with: A. d’Aspremont
Options

Options are time bombs. —Warren Buffett, 2003

let \( x(T) \) denote the price of an asset at time \( T \)

an *European call option* with maturity \( T \) and strike price \( K \) is a contract with payoff

\[
(x(T) - K)_+ = \sup(x(T) - K, 0)
\]
Options

*Options are time bombs.* —Warren Buffett, 2003

Let \( x(T) \) denote the price of an asset at time \( T \)

an *European call option* with maturity \( T \) and strike price \( K \) is a contract with payoff

\[
(x(T) - K)_+ = \sup(x(T) - K, 0)
\]

**what is the price of the option,**

**assuming no arbitrage ("free lunch") is possible?**
fundamental result of finance: assuming a zero risk-free interest rate, the no-arbitrage price of the option is
\[ E_\pi(x(T) - K)_+ \]
for some distribution \( \pi \) on the asset price at time \( T \)

such a \( \pi \) is called risk-neutral
Forwards

under the risk-neutral distribution \( \pi \), the current asset price is

\[ E_\pi x(T) \]

i.e., \( \pi \) is a martingale
Basket options

let $x(T)$ be a $n$-vector of prices of assets at time $T$

a basket option with weight vector $w$ and strike $K$ is the contract with payoff

$$(w^T x(T) - K)_+$$

denote the basket option by $(w, K)$ and its price by

$$C_\pi(w, K) := \mathbf{E}_\pi(w^T x(T) - K)_+$$

(maturity date is implicit here)
Option pricing problem

given

- $w_0, w_1, \ldots, w_m$ in $\mathbb{R}_+^n$ (basket weights)
- $K_0, K_1, \ldots, K_m$ in $\mathbb{R}_+$ (strike prices)
- $p_1, \ldots, p_m$ in $\mathbb{R}_+$ (observed option prices)

determine the price of the basket option with weight $w_0$ and strike price $K_0$
Option pricing problem

given

- $w_0, w_1, \ldots, w_m$ in $\mathbb{R}_+^n$ (basket weights)
- $K_0, K_1, \ldots, K_m$ in $\mathbb{R}_+$ (strike prices)
- $p_1, \ldots, p_m$ in $\mathbb{R}_+$ (observed option prices)

determine the price of the basket option with weight $w_0$ and strike price $K_0$

in practice, we are also given the current asset prices themselves, $q \in \mathbb{R}_+^n$ ("forwards")
Challenges & methods

challenges:

- the risk-neutral measure is not the empirical distribution of assets
- hence, we may have to rely on option & forward prices only

two approaches:

- model-based approach
- arbitrage-based approach
Model-based approach

• assume a log-normal diffusion model for the asset prices

\[ ds = Asdt + Bsdw, \]

where \( s = \log x \), and \( w \) is a multidimensional Brownian motion

• fit the model to observed option prices
Model-based approach

• assume a log-normal diffusion model for the asset prices

\[ ds = Asdt + Bsdw, \]

where \( s = \log x \), and \( w \) is a multidimensional Brownian motion

• fit the model to observed option prices

with some approximations, this problem is an SDP (Aspremont, 2000)
Model-based approach

• assume a log-normal diffusion model for the asset prices

\[ ds = Asdt + Bsdw, \]

where \( s = \log x \), and \( w \) is a multidimensional Brownian motion

• fit the model to observed option prices

pros and cons:

• very versatile, and ”easy” to solve (albeit only recently)

• makes a structural assumption about the risk-neutral measure \( \pi \)

• provides a point estimate for the price of basket \((w_0, K_0)\)
No-arbitrage approach

find \textbf{bounds} on the price of basket \((w_0, K_0)\) by solving semi-infinite LP

\[
\sup_\pi / \inf_\pi \quad \mathbb{E}_\pi(w_0^T x - K_0)_+
\]

s.t. \quad \mathbb{E}_\pi(w_i^T x - K_i)_+ = p_i, \quad i = 1, \ldots, m

the optimization variable is the risk-neutral probability measure \(\pi\)
No-arbitrage approach

find bounds on the price of basket \((w_0, K_0)\) by solving semi-infinite LP

\[
\sup_{\pi} / \inf_{\pi} \ E_\pi (w_0^T x - K_0)_+ \\
\text{s.t.} \quad E_\pi (w_i^T x - K_i)_+ = p_i, \ i = 1, \ldots, m
\]

the optimization variable is the risk-neutral probability measure \(\pi\)

pros and cons:

- problem may be difficult
- approach provides only bounds, but . . .
- . . . makes no assumptions about market dynamics
Upper bound problem

upper bound problem:

\[
p_{\text{sup}} := \sup_{\pi \in \mathcal{P}} \int_{\Omega} \phi_0(x) \pi(x) \, dx : \int_{\Omega} \pi(x) \, dx = 1,
\]

\[
\int_{\Omega} \phi_i(x) \pi(x) \, dx = p_i, \ i = 1, \ldots, m,
\]

where

- \( \Omega = \mathbb{R}^n_+ \)
- \( \mathcal{P} \) is the set of densities with support in \( \Omega \)
- \( \phi_i(x) := (w_i^T x - K_i)_+ \), \( i = 0, 1, \ldots, m \)
Upper bound problem

upper bound problem:

\[ p^{\text{sup}} := \sup_{f \in \mathcal{F}} \int_{\Omega} \phi_0(x)f(x)dx : \int_{\Omega} f(x)dx = 1, \]
\[ \int_{\Omega} \phi_i(x)f(x)dx = p_i, \quad i = 1, \ldots, m, \]

Lagrangian:

\[ \mathcal{L}(\pi, \lambda, \lambda_0) = \int_{\Omega} \phi_0(x)\pi(x)dx + \lambda_0 \left( 1 - \int_{\Omega} \pi(x)dx \right) \]
\[ + \lambda^T \left( p - \int_{\Omega} \phi(x)\pi(x)dx \right) \]
Upper bound problem

upper bound problem:

\[ p^{\text{sup}} := \sup_{f \in \mathcal{F}} \int_{\Omega} \phi_0(x) f(x) \, dx : \int_{\Omega} f(x) \, dx = 1, \]
\[ \int_{\Omega} \phi_i(x) f(x) \, dx = p_i, \quad i = 1, \ldots, m, \]

the dual is the robust linear programming problem

\[ d^{\text{sup}} := \inf_{\lambda \in \mathbb{R}^m} \lambda^T p + \lambda_0 \]
\[ \text{s.t.} \quad \forall x \in \Omega, \quad \lambda^T \phi(x) + \lambda_0 \geq \phi_0(x) \]
Dual gives a hedging strategy

let $\lambda_0, \lambda$ be feasible for the dual:

$$d^{sup} := \inf_{\lambda \in \mathbb{R}^m} \lambda^T p + \lambda_0$$

s.t. $\forall x \in \Omega, \lambda^T \phi(x) + \lambda_0 \geq \phi_0(x)$ \quad (*)$

strategy: invest $\lambda_i$ in basket $(w_i, K_i)$, $\lambda_0$ in cash
Dual gives a hedging strategy

let \( \lambda_0, \lambda \) be feasible for the dual:

\[
\begin{align*}
  d_{\text{sup}} := & \inf_{\lambda \in \mathbb{R}^m} \lambda^T p + \lambda_0 \\
  \text{s.t.} & \quad \forall x \in \Omega, \quad \lambda^T \phi(x) + \lambda_0 \geq \phi_0(x) \\
\end{align*}
\]

strategy: invest \( \lambda_i \) in basket \((w_i, K_i)\), \( \lambda_0 \) in cash

- price of strategy: \( \lambda^T p + \lambda_0 \)
- taking expectations in \((*)\), get

\[
\lambda^T p + \lambda_0 \geq \mathbb{E}_\pi \phi_0(x) = p_{\text{sup}}
\]
(proves weak duality)
A special case

we are given option prices on the \( m = n \) individual assets, as well as forward prices

\[
\min_{\pi} / \sup_{\pi} \quad E_\pi (w_0^T x - K_0)_+ \\
\text{s.t.} \quad E_\pi (x_i - K_i)_+ = p_i, \quad i = 1, \ldots, n \\
E_\pi x = q
\]

we may further relax the problem by ignoring the forward price information
Special case: upper bound

- **no-arbitrage**: problem is feasible iff $0 \leq p \leq q \leq p + K$
- upper bound is given by

$$d^\text{sup} = \max_{0 \leq j \leq n+1} w_0^T p + \sum_i w_{0,i} \min(q_i - p_i, \beta_j K_i) - \beta_j K_0,$$

where $\beta_0 = 0 \leq \beta_j := (q_j - p_j) / K_j \leq 1 = \beta_{n+1}$

- upper bound is attained, hence $d^\text{sup} = p^\text{sup}$
Ignoring forward prices

ignore the constraints

$$E_\pi x = q$$

obtain formula by maximizing $d^{\text{sup}}$ wrt $q$:

$$d^{\text{sup}} = p^{\text{sup}} = w_0^T p + (w_0^T K - K_0)_+$$
Ignoring forward prices

ignore the constraints

\[ E_\pi x = q \]

obtain formula by maximizing \( d^{\text{sup}} \) wrt \( q \):

\[ d^{\text{sup}} = p^{\text{sup}} = w_0^T p + (w_0^T K - K_0)_+ \]

makes sense:

- concave in \( p \) (the RHS of our primal LP)
- convex in \( (w_0, K_0) \)
- interpolates \( p_i \) at \( w_0 = i\)-th unit vector of \( \mathbb{R}^n \), and \( K_0 = K_i \)
Sketch of proof

weak duality follows from homogeneity and convexity of $x \rightarrow x_+$:

\[
E_\pi(w_0^T x - K_0)_+ = E_\pi(w_0^T (x - K) + (w_0^T K - K_0))_+ \\
\leq w_0^T E_\pi(x - K)_+ + (w_0^T K - K_0)_+ \quad (w_0 \geq 0) \\
= w_0^T p + (w_0^T K - K_0)_+
\]
Sketch of proof

weak duality follows from homogeneity and convexity of $x \rightarrow x_+$:

$$
\mathbb{E}_\pi(w_0^T x - K_0)_+ = \mathbb{E}_\pi(w_0^T(x - K) + (w_0^T K - K_0))_+
\leq w_0^T \mathbb{E}_\pi(x - K)_+ + (w_0^T K - K_0)_+ \quad (w_0 \geq 0)
= w_0^T p + (w_0^T K - K_0)_+
$$

strong duality: choose $x = p + K$ with pty 1 if $w_0^T K \geq K_0$, otherwise take a limit of feasible distributions

$$
x = \begin{cases} 
\epsilon^{-1} p + K & \text{with probability } \epsilon, \\
0 & \text{with probability } 1 - \epsilon.
\end{cases}
$$
Lower bound

• dual problem reduces to a finite LP
  (with $O(n)$ variables and constraints)
Lower bound

• dual problem reduces to a finite LP
  (with $O(n)$ variables and constraints)

• if we ignore forward prices

\[
p^{\text{inf}} = d^{\text{inf}} = \sum_{i : K_i w_i \geq K_0} p_i w_i
  + \max_{j : K_j w_j < K_0} \left( \sum_{i : K_i w_i < K_0} p_i w_i \min(1, \frac{K_0 - K_j w_j}{K_0 - K_i w_i}) - K_0 + w_j K_j \right)
\]

in which case, direct proof of perfect duality
General case: integral transform approach

the option price function

\[ C(w, K) = \mathbb{E}_\pi (w^T x - K)_+ \]

is an integral transform of measure \( \pi \), with kernel the payoff function

\( (w^T x - K)_+ \)
**General case: integral transform approach**

the option price function

\[ C(w, K) = \mathbb{E}_\pi (w^T x - K)_+ \quad (1) \]

is an integral transform of measure \( \pi \), with kernel the payoff function \((w^T x - K)_+\).

make \( C \) the variable, and solve interpolation problem

\[
\sup_C \frac{\inf_C C(w_0, K_0)}{C(w_i, K_i) = p_i, \ i = 1, \ldots, m, \ C \text{ of the form } (1)}
\]
LP relaxation

conditions under which $C$ is of form

$$C(w, K) = E_{\pi}(w^T x - K)_+$$

for some measure $\pi$ exist, but seem hard to check

semi-infinite LP relaxation:

$$\sup_C \inf_C C(w_0, K_0)$$

s.t.

$C(w, K)$ convex in $(K, w)$

$C(w, K)$ homogeneous of degree 1

$-1 \leq \partial C(w, K)/\partial K \leq 0$ and $C(w, K)$ nondecreasing in $w$

$C(w_i, K_i) = p_i$, $i = 1, \ldots, m$
Finite LP formulation

optimal $C$ of semi-infinite LP is piecewise affine

hence the semi-infinite LP can be reduced exactly to a finite LP

$$\sup / \inf \quad p_0$$

subject to

$$\langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i, \quad i, j = 0, \ldots, m + n + 1$$

$$g_{i,j} \geq 0, -1 \leq g_{i,n+1} \leq 0, \quad i = 0, \ldots, m + n + 1, \quad j = 1, \ldots, n$$

$$\langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 0, \ldots, m + n + 1,$$

where the variables $g_i$ are subgradients of $C^{\text{opt}}$

(for upper bound, in special case, LP is exact)
Robustness

in practice we have uncertainty:

- basket weights may change, or not exactly known at present time
- price information may be noisy (e.g., bid-ask)
- strike prices also may vary
- hedging strategies may be implemented with errors

we address the upper bound problem, in the special case
Bid-ask spread

bid-ask spread corresponds to a box uncertainty for the vector of observed option prices

\[ \underline{p} \leq p \leq \bar{p} \]

the worst-case value of upper (lower) bound is attained at \( p^{\text{worst}} = \underline{p} \)

(or \( p^{\text{worst}} = \bar{p} \))

we may want to introduce "correlation" in the model . . . ellipsoidal uncertainty on \( p \) is as easy
Ellipsoidal uncertainty in basket weights

\textbf{worst-case} upper bound under weight uncertainty is

\[
p^{\text{sup}} = \max_{w \in \mathcal{E}} w^T p + (w^T K - K_0)_+
\]

where \( \mathcal{E} = \{ \hat{w} + Ru : \|u\|_2 \leq 1 \} \) is a given ellipsoid

we have

\[
p^{\text{sup}} = \max_{w,t} w^T p + t(w^T K - K_0) : w \in \mathcal{E}, \ 0 \leq t \leq 1
\]

\[
= \max_{t=0,1} (tK + p)^T w + \|R^T (tK + p)\|_2 - tK_0
\]
Example

Price bounds on a basket call option

- Simulated price
- Upper bound (explicit)
- Lower bound (explicit LP)
- Upper bound (LP relax.)
- Lower bound (LP relax.)

Strike $K_0$

Price
Robustness for the general case

i have no idea about this
Summary

- general problem important in practice
- special cases yields easy-to-compute bounds
- developed an LP relaxation for general case
- relaxation exact in some special cases
Further research

• imposing smoothness constraints
• multi-period problem (Bertsimas, 2003)
• link with model-based approaches
Further research

- imposing smoothness constraints
- multi-period problem (Bertsimas, 2003)
- link with model-based approaches