Ellipsoidal Bounds for Uncertain Linear Equations and Dynamical Systems*

Giuseppe Calafiore† and Laurent El Ghaoui‡

†Dipartimento di Automatica e Informatica
Politecnico di Torino
Corso Duca degli Abruzzi, 24 – 10129 Torino, Italy

‡Dept. of Electrical Engineering and Computer Science
University of California at Berkeley, USA

July 24, 2003

Abstract

In this paper, we discuss semidefinite relaxation techniques for computing minimal size ellipsoids that bound the solution set of a system of uncertain linear equations. The proposed technique is based on the combination of a quadratic embedding of the uncertainty, and the $\mathcal{S}$-procedure. This formulation leads to convex optimization problems that can be essentially solved in $O(n^3)$ — n being the size of unknown vector — by means of suitable interior point barrier methods, as well as to closed form results in some particular cases. We further show that the uncertain linear equations paradigm can be directly applied to various state-bounding problems for dynamical systems subject to set-valued noise and model uncertainty.

Keywords: Uncertain linear equations, set-valued filtering, interior-point methods.

1 Introduction

This paper discusses a technique for computing deterministic confidence bounds on the solutions of systems of linear equations whose coefficients are imprecisely known, and presents an application of this technique to the problem of set-valued prediction and filtering for uncertain dynamical systems.

Uncertain linear equations (ULE) arise in many engineering endeavors, when the actual data are imprecisely known and reliable bounds on the possible solutions need to be determined. For instance, in many problems of system identification one must solve a linear system of normal equations arising from minimization of a least-squares criterion. When the regression data are subject to bounded uncertainty, this gives rise to a system of uncertain linear equations of the type examined in this paper. Similarly, ULEs arise in Vandermonde systems for polynomial interpolation, when the abscissae of the interpolation points are assumed uncertain, as well as in Toeplitz systems for finite impulse response estimation (see an example in Section 2.4.2). Also, in solid and structural

*This work has been supported in part by FIRB funds of Italian Ministry of University and Research.

mechanics, uncertain linear equations are used to determine bounds on the system dynamic response for many scenarios of load and stiffness terms, [19, 23]. Specific applications in the context of set-valued prediction and filtering for uncertain dynamical systems are discussed in Section 3 of this paper.

A basic version of the problem we deal with is well-known in the context of interval linear algebra, where one is given matrices $A \in \mathbb{R}^{n,n}$ and $y \in \mathbb{R}^n$, the elements of which are only known within intervals, and seeks to compute intervals of confidence for the set of solutions, if any, to the equation $Ax = y$, see e.g. [15, 27]. Obtaining exact estimates on the confidence intervals for the elements of x in the above context is known to be an NP-hard problem, [33, 34].

Here, we consider a more general situation in which the data matrix $[A \;\; y]$ belongs to an uncertainty set $\mathcal{U}$ described by means of a linear fractional representation (LFR), and use semidefinite relaxation techniques [13] to determine efficiently computable minimal ellipsoidal bounds for the set of solutions. In particular, we develop a special decoupled formulation of the problem which leads to very efficient interior-point algorithms that scale with problem size essentially as $O(n^3)$, see Section 2.1 and Section 2.2. Besides, we discuss special situations in which semidefinite relaxations are lossless, and show how we can recover explicit closed-form solutions in these cases. Semidefinite relaxation techniques for uncertain linear equations have been originally introduced by the authors in [7].

In a subsequent part of the paper, we show the versatility of the ULE model by applying it to the solution of set-valued prediction and filtering problems relative to uncertain, discrete-time dynamical systems. The problem of determining a set that is guaranteed to contain the state of the system, despite the action of unknown-but-bounded additive noise, has a long history in the control literature. Early references on this topic include [3, 10, 35, 36], while more recent contributions are found in [6, 20, 28, 37], to mention but a few. A fundamental point to remark is that in all the above mentioned references the *system description* is assumed to be exactly known, while the main contribution of this paper is to derive efficiently computable bounds on the system states when, in addition to unknown-but-bounded additive noise, structured deterministic uncertainty affects the system description in a possibly non-linear fashion. Semidefinite relaxation techniques in this context have been first introduced by the authors in [8] for set-valued simulation, and in [12] for set-valued filtering. In the present paper, we derive similar results for predictor/corrector filter recursions, using the unifying theoretical framework provided by the ULE paradigm.

This paper is organized as follows. Section 2 introduces the ULE model, and contains all the relative fundamental results. Section 2.2 provides a detailed discussion on the numerical complexity of computing bounds on ULEs, while Section 2.3 presents closed form results in the special case of unstructured uncertainty. Numerical examples are discussed in Section 2.4. Finally, Section 3 discusses the application of the ULE model in set-valued prediction and filtering for uncertain dynamical systems. A numerical example related to set-valued filtering is presented in Section 3.3, while conclusions are drawn in Section 4. To improve readability, several useful technical results have been collected in the appendices.

1.1 Notation and preliminaries

For a square matrix X, $X \succ 0$ (resp. $X \succeq 0$) means X is symmetric, and positive-definite (resp. semidefinite). For a matrix $X \in \mathbb{R}^{n,m}$, $\mathcal{R}(X)$ denotes the space generated by the columns of X, and $\mathcal{N}(X)$ denotes the kernel of X. An orthogonal complement of X is denoted by $X_\perp$, which is a matrix containing by columns a basis for $\mathcal{N}(X)$, i.e. a matrix of maximal rank such that $XX_\perp = 0$. $X^\dagger$ denotes the (Moore-Penrose) pseudo-inverse of X. $\|X\|$ denotes the spectral (maximum singular value) norm of X, or the standard Euclidean norm, in case of vectors. I_n, $0_{n,m}$, and $1_{n,m}$ denote respectively the identity matrix of dimension $n \times n$, the zero matrix, and the matrix of ones of dimension $n \times m$; dimensions are sometimes omitted when they can be easily inferred from context.

Ellipsoids. Ellipsoids will be described as

$$\mathcal{E} = \{x : x = \hat{x} + Ez, \|z\| \leq 1\},$$

where $\hat{x} \in \mathbb{R}^n$ is the center, and $E \in \mathbb{R}^{n,m}$, $\mathrm{Rank}(E) = m \leq n$ is the *shape matrix* of the ellipsoid. This representation can handle all bounded ellipsoids, including 'flat' ellipsoids, such as points or intervals. An alternative description involves the squared shape matrix $P = EE^T$

$$\mathcal{E}(P, \hat{x}) = \left\{ x : \begin{bmatrix} P & (x - \hat{x}) \\ (x - \hat{x})^T & 1 \end{bmatrix} \succeq 0 \right\}.$$

When $P \succ 0$, the previous expression is also equivalent to

$$\mathcal{E}(P, \hat{x}) = \left\{ x : (x - \hat{x})^T P^{-1} (x - \hat{x}) \leq 1 \right\}.$$

The 'size' of an ellipsoid is a function of the squared shape matrix P, and will be denoted $f(P)$. Throughout this paper, $f(P)$ will be either $\mathrm{trace}\,(P)$, which corresponds to the sum of squares of the semi-axes lengths, or $\log\det(P)$, which is related to the volume of the ellipsoid.

Uncertainty description. Structured uncertainty is described as follows: $\mathbf{\Delta}$ is a subspace of $\mathbb{R}^{n_p, n_q}$, called the *structure subspace* (for instance, the space of matrices with certain block-diagonal structure). Then, the uncertain matrix Δ is restricted to

$$\Delta \in \mathbf{\Delta}_1 \doteq \{\Delta \in \mathbf{\Delta} : \|\Delta\| \leq 1\}.$$

Associated to the structure subspace, we introduce the *scaling subspace* $\mathcal{B}(\mathbf{\Delta})$

$$\mathcal{B}(\mathbf{\Delta}) = \left\{ (S, T, G) \ : \ \forall \Delta \in \mathbf{\Delta},\ S\Delta = \Delta T,\ G\Delta = -\Delta^T G^T \right\}. \tag{1.1}$$

A structure that frequently arises in practice is the *independent block-diagonal* structure

$$\mathbf{\Delta} = \{\Delta : \Delta = \mathrm{diag}\,(\Delta_1, \ldots, \Delta_\ell),\ \Delta_i \in \mathbb{R}^{n_{pi}, n_{qi}}\}. \tag{1.2}$$

For this structure, the scaling subspace is constituted of all triples S, T, G with $S = \mathrm{diag}\,(\lambda_1 I_{n_{p1}}, \ldots, \lambda_\ell I_{n_{p\ell}})$, $T = \mathrm{diag}\,(\lambda_1 I_{n_{q1}}, \ldots, \lambda_\ell I_{n_{q\ell}})$, $G = 0$. A particular case of this situation arises for $\ell = 1$, and it is denoted as the *unstructured uncertainty* case.

Independent scalar uncertain parameters $\delta_1, \ldots, \delta_\ell$ with bounded magnitude $|\delta_i| \leq 1$ are represented in our framework via the structure subspace

$$\mathbf{\Delta} = \left\{ \Delta : \Delta = \mathrm{diag}\,(\delta_1 I_{n_{p1}}, \ldots, \delta_\ell I_{n_{p\ell}}),\ \delta_i \in \mathbb{R} \right\}, \tag{1.3}$$

and the corresponding scaling subspace, constituted of all triples S, T, G with $S = T = \mathrm{diag}\,(S_1 \ldots, S_\ell)$, $S_i = S_i^T \in \mathbb{R}^{n_{pi},n_{pi}}$, $G = \mathrm{diag}\,(G_1, \ldots, G_\ell)$, $G_i = -G_i^T \in \mathbb{R}^{n_{pi},n_{pi}}$.

More general uncertainty structures, together with their corresponding scaling spaces, are detailed for instance in [11, 13].

2 Uncertain Linear Equations

Let the uncertain data be described as

$$[A(\Delta) \;\; y(\Delta)] = [A \;\; y] + L\Delta(I - H\Delta)^{-1}[R_A \;\; R_y], \tag{2.4}$$

where $A \in \mathbb{R}^{m,n}$, $y \in \mathbb{R}^m$, $L \in \mathbb{R}^{m,n_p}$, $R_A \in \mathbb{R}^{n_q,n}$, $R_y \in \mathbb{R}^{n_q}$, $H \in \mathbb{R}^{n_q,n_p}$, and $\Delta \in \mathbf{\Delta}_1 \subset \mathbb{R}^{n_p,n_q}$, and let this linear fractional representation (LFR) be well-posed over $\mathbf{\Delta}_1$, meaning that $\det(I - H\Delta) \neq 0$, $\forall \Delta \in \mathbf{\Delta}_1$. Lemma A.2 reported in the Appendix provides a well-known and readily checkable sufficient condition for the well-posedness of the above linear fractional representation.

The representation (2.4) includes as special cases, for instance, interval matrices discussed in many references [15, 27, 33, 34], as well as additive uncertainty of the form $[A + \Delta_A \;\; y + \Delta_y]$. In this latter case, the linear fractional representation is simply given by $L = [I_m \;\; I_m]$, $[R_A \;\; R_y] = I_{n+1}$, $H = 0_{n+1,2m}$, and $\Delta = \mathrm{diag}\,(\Delta_A, \Delta_y)$. The description (2.4) also allows for representation of general rational matrix functions of a vector of uncertain parameters $\delta_1, \ldots, \delta_\ell$, see [11, 13] for further details and constructive procedure for building the LFR in this general case.

Define now the set $\mathcal{X}$ of all the possible solutions to the linear equations $A(\Delta)x = y(\Delta)$, i.e.

$$\mathcal{X} \doteq \{x : \; A(\Delta)x = y(\Delta), \text{ for some } \Delta \in \mathbf{\Delta}_1\}.$$

In the sequel, we provide conditions under which the set $\mathcal{X}$ is contained in a bounded ellipsoid $\mathcal{E}$, and we exploit these conditions to determine a minimal (in the sense of the selected size measure $f(\cdot)$) ellipsoid containing the solution set $\mathcal{X}$.

We first state a key lemma.

Lemma 2.1. *Let*

$$\Psi \doteq [A \;\; L \;\; y], \tag{2.5}$$

$$\Upsilon \doteq \begin{bmatrix} R_A & H & R_y \\ 0_{n_p,n} & I_{n_p} & 0_{n_p,1} \end{bmatrix}, \tag{2.6}$$

$$\Omega(S,T,G) \doteq \Upsilon^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \Upsilon. \tag{2.7}$$

Let further the orthogonal complement $\Psi_\perp$ be chosen as

$$\Psi_\perp \doteq \left[\begin{array}{c|c} \Psi_{\perp 1} & \psi_{\perp 2} \\ \hline 0 \cdots 0 & -1 \end{array}\right], \tag{2.8}$$

where $\Psi_{\perp 1}$ is an orthogonal complement of $[A \;\; L]$, and $\psi_{\perp 2}$ is any vector such that $[A \;\; L]\psi_{\perp 2} = y$. (If no such $\psi_{\perp 2}$ exists, then the solution set $\mathcal{X}$ is empty).

Then, for any triple $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, with $S \succeq 0$ and $T \succeq 0$, the set

$$\overline{\mathcal{X}}_{S,T,G} \doteq \left\{ x = [I_n\ 0\ 0]\Psi_\perp \left[\begin{array}{c} \nu \\ 1 \end{array} \right], \nu :\ [\nu^T\ \ 1]\Psi_\perp^T \Omega(S,T,G)\Psi_\perp [\nu^T\ \ 1]^T \geq 0 \right\}, \tag{2.9}$$

is an outer approximation for the solution set $\mathcal{X}$, i.e. $\mathcal{X} \subseteq \overline{\mathcal{X}}_{S,T,G}$.

Furthermore, when $\mathbf{\Delta}$ is a full block (unstructured uncertainty) the approximation is exact, i.e. $\overline{\mathcal{X}}_{S,T,G} \equiv \mathcal{X}$. In this latter case, the solution set is the quadratic *set*

$$\mathcal{X} = \left\{ x = [I_n\ 0\ 0]\Psi_\perp \left[\begin{array}{c} \nu \\ 1 \end{array} \right], \nu :\ [\nu^T\ \ 1]\Psi_\perp^T \Omega(I,I,0)\Psi_\perp [\nu^T\ \ 1]^T \geq 0 \right\}. \tag{2.10}$$

$\star$

Proof. Consider the linear fractional description (2.4), and rewrite equation $A(\Delta)x = y(\Delta)$ as

$$Ax - y + L\Delta(I - H\Delta)^{-1}(R_A x - R_y) = 0,$$

which in turn can be expressed using a slack vector p in the form

$$\begin{aligned} Ax - y + Lp &= 0 && (2.11)\\ R_A x + Hp - R_y &= q && (2.12)\\ p &= \Delta q. && (2.13) \end{aligned}$$

Let Ψ be as defined in (2.5), and let

$$\xi \doteq [x^T\ \ p^T\ \ -1]^T, \tag{2.14}$$

then all vectors ξ compatible with (2.11) must be orthogonal to Ψ, and can be expressed as

$$\xi = \Psi_\perp \eta, \text{ with } \eta \doteq \left[\begin{array}{c} \nu \\ 1 \end{array} \right],\ \Psi_\perp \doteq \left[\begin{array}{c|c} \Psi_{\perp 1} & \psi_{\perp 2} \\ \hline 0 \cdots 0 & -1 \end{array} \right], \tag{2.15}$$

where $\Psi_{\perp 1}$ is an orthogonal complement of $[A\ L]$, and $\psi_{\perp 2}$ is any vector such that $[A\ L]\psi_{\perp 2} = y$. Notice that if no such $\psi_{\perp 2}$ exists, then (2.11) is not solvable, and hence the solution set $\mathcal{X}$ is clearly empty. All feasible ξ must therefore lie on the flat

$$\mathcal{F} \doteq \left\{ \xi :\ \xi = \Psi_\perp \eta, \text{ with } \eta = \left[\begin{array}{cc} \nu^T & 1 \end{array} \right]^T \right\},$$

and the corresponding feasible x on the projection $\mathcal{F}_x \doteq \{x = [I_n\ 0_{n,n_p}\ 0_{n,1}]\xi :\ \xi \in \mathcal{F}\}$. The feasible ξ are further constrained by (2.12)–(2.13): By the Quadratic Embedding Lemma A.5 (in the Appendix), for any triple $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$, the set of all pairs (q,p) such that $p = \Delta q$ for some $\Delta \in \mathbf{\Delta}_1$, is bounded by the set

$$\mathcal{Q}_{S,T,G} \doteq \left\{ \left[\begin{array}{c} q \\ p \end{array} \right] :\ \left[\begin{array}{c} q \\ p \end{array} \right]^T \left[\begin{array}{cc} T & G \\ G^T & -S \end{array} \right] \left[\begin{array}{c} q \\ p \end{array} \right] \geq 0 \right\}. \tag{2.16}$$

Therefore, the set of ξ compatible with (2.12)–(2.13) is bounded by the set

$$\mathcal{H}_{S,T,G} \doteq \{\xi :\ \xi^T \Omega(S,T,G)\xi \geq 0\}, \tag{2.17}$$

where $\Omega(S,T,G)$ is defined in (2.7). To conclude, the set of ξ compatible with all conditions (2.11)–(2.13) is bounded by the intersection $\mathcal{F} \cap \mathcal{H}_{S,T,G}$, and therefore $\mathcal{X} \subseteq \overline{\mathcal{X}}_{S,T,G}$, where $\overline{\mathcal{X}}_{S,T,G}$ is the projection

$$\overline{\mathcal{X}}_{S,T,G} = \left\{x = [I_n\ 0\ 0]\Psi_\perp \eta :\ \eta^T \Psi_\perp^T \Omega(S,T,G)\Psi_\perp \eta \geq 0\right\}, \tag{2.18}$$

with η and $\Psi_\perp$ defined in (2.15).

When $\mathbf{\Delta}$ is unstructured, the embedding in Lemma A.5 is tight, and the approximation is exact, i.e. $\overline{\mathcal{X}}_{S,T,G} = \mathcal{X}$. Moreover, in this case the scalings are $S = \lambda I$, $T = \lambda I$, $G = 0$, and hence (2.10) follows. □

The next theorem provides a characterization of a bounding ellipsoid for the solution set $\mathcal{X}$.

Theorem 2.1. *Let all symbols be defined as in Lemma 2.1. If there exist $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$ such that*

$$\left[\begin{array}{c|c} P & [I\ \ 0\ \ \hat{x}]\Psi_\perp \\ \hline \Psi_\perp^T [I\ \ 0\ \ \hat{x}]^T & \Psi_\perp^T \left(\mathrm{diag}\,(0,0,1) - \Omega(S,T,G)\right)\Psi_\perp \end{array}\right] \succeq 0 \tag{2.19}$$

is feasible, then the ellipsoid $\mathcal{E}(P,\hat{x})$ contains the solution set $\mathcal{X}$. ⋆

Proof. From Lemma 2.1, we have that for any triple $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$, the condition $\mathcal{E}(P,\hat{x}) \supseteq \overline{\mathcal{X}}_{S,T,G}$ implies $\mathcal{E}(P,\hat{x}) \supseteq \mathcal{X}$. Consider then the following points.

1. The family of ellipsoids $\mathcal{E}(P,\hat{x})$ that lie in $\mathcal{F}_x$ satisfy the flatness condition $(I - P^\dagger P)(x - \hat{x}) = 0,\ \forall x \in \mathcal{F}_x$, which can be expressed using the notation introduced previously, as $(I - P^\dagger P)[I_n\ 0\ \hat{x}]\Psi_\perp \eta = 0,\ \forall \eta$, i.e.

$$(I - P^\dagger P)[I_n\ 0\ \hat{x}]\Psi_\perp = 0. \tag{2.20}$$

2. An ellipsoid $\mathcal{E}(P,\hat{x}) \subset \mathcal{F}_x$ contains the point $x = [I_n\ 0\ 0]\Psi_\perp \eta \in \mathcal{F}_x$ if and only if (notice that $x - \hat{x} = [I_n\ 0\ \hat{x}]\Psi_\perp \eta$)

$$\left[\begin{array}{cc} P & [I_n\ 0\ \hat{x}]\Psi_\perp \eta \\ * & 1 \end{array}\right] \succeq 0. \tag{2.21}$$

Using Schur complements (Lemma A.6 in the Appendix), this is rewritten as

$$\begin{aligned} 1 - \eta^T \Psi_\perp^T [I_n\ 0\ \hat{x}]^T P^\dagger [I_n\ 0\ \hat{x}]\Psi_\perp \eta &\geq 0 && (2.22)\\ (I - P^\dagger P)[I_n\ 0\ \hat{x}]\Psi_\perp \eta &= 0. && (2.23) \end{aligned}$$

Since (2.20) holds for all ellipsoids that lie entirely in $\mathcal{F}_x$, condition (2.23) is always satisfied, therefore the ellipsoid $\mathcal{E}(P,\hat{x}) \subset \mathcal{F}_x$ contains the point $x = [I_n\ 0\ 0]\Psi_\perp \eta \in \mathcal{F}_x$ if and only if (2.22) is satisfied.

3. The ellipsoid $\mathcal{E}(P,\hat{x})$ lies in $\mathcal{F}_x$ and contains $\overline{\mathcal{X}}_{S,T,G}$ if and only if (2.20) holds, and (2.22) is satisfied for all η such that $\eta^T\Psi_\perp^T\Omega(S,T,G)\Psi_\perp\eta \geq 0$. By the $\mathcal{S}$-procedure and homogenization (see Lemma A.3 and Lemma A.4), the above happens if (2.20) holds, and there exist $\tau \geq 0$ such that

$$\Psi_\perp^T\left(\mathrm{diag}\,(0,0,1) - [I\ \ 0\ \ \hat{x}]^T P^\dagger [I\ \ 0\ \ \hat{x}]\right)\Psi_\perp \succeq \tau\Psi_\perp^T\Omega(S,T,G)\Psi_\perp.$$

Using the Schur complement rule, the two previous conditions are written in the equivalent matrix inequality form as

$$\left[\begin{array}{c|c} P & [I\ \ 0\ \ \hat{x}]\Psi_\perp \\ \hline \Psi_\perp^T[I\ \ 0\ \ \hat{x}]^T & \Psi_\perp^T\left(\mathrm{diag}\,(0,0,1) - \tau\Omega(S,T,G)\right)\Psi_\perp \end{array}\right] \succeq 0. \tag{2.24}$$

Further, from Lemma A.4, we have that (2.24) is also a necessary condition for the inclusion, provided that there exist η_0: $\eta_0^T\Psi_\perp^T\Omega(S,T,G)\Psi_\perp\eta_0 > 0$.

In synthesis, if there exist $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$, such that (2.24) is satisfied (notice that τ can be absorbed in the S,T,G variables and then eliminated from the condition), then the ellipsoid $\mathcal{E}(P,\hat{x})$ lies in $\mathcal{F}_x$ and contains $\mathcal{X}$. Moreover, if there exist η_0: $\eta_0^T\Psi_\perp^T\Omega(S,T,G)\Psi_\perp\eta_0 > 0$, (2.24) is also necessary for an ellipsoid $\mathcal{E}(P,\hat{x}) \subset \mathcal{F}_x$ to include $\overline{\mathcal{X}}_{S,T,G}$. □

Remark 2.1. Based on the condition (2.19), we can subsequently minimize a (convex) size measure $f(P)$ of the bounding ellipsoid, subject to this inclusion constraint. Solving the convex optimization problem in the variables $P,\hat{x},S,T,G$

$$\text{minimize } f(P) \text{ subject to} \tag{2.25}$$

$$(S,T,G) \in \mathcal{B}(\mathbf{\Delta}),\ S \succeq 0, T \succeq 0, (2.19) \tag{2.26}$$

hence yields an outer ellipsoidal approximation of $\mathcal{X}$, that is optimal in the sense of the sufficient condition (2.19). Notice that this optimization problem is a semidefinite program (SDP), if $f(P) = \mathrm{trace}\,(P)$, and a MAXDET problem, if $f(P) = \log\det(P)$. In both cases the problem can be efficiently solved in polynomial-time by interior point methods for convex programming, [39, 40].

We also remark that Lemma 2.1 can be used for directly determining optimized bounds on individual elements of the solution vector x. In this case, one is not interested in determining a bounding ellipsoid for the entire solution vector, but rather a minimal width interval containing a specific entry of x. This is basically a special case of the problem considered in Theorem 2.1, and we leave this easy modification to the reader. ◀

In the particular case of unstructured uncertainty, the condition expressed in the Theorem 2.1 becomes necessary and sufficient, as detailed in the following corollary.

Corollary 2.1. *Let $\mathbf{\Delta} = \mathbb{R}^{n_p,n_q}$, and assume there exists η_0 such that*

$$\eta_0^T\Psi_\perp^T\Upsilon^T\left[\begin{array}{cc} I & 0 \\ 0 & -I \end{array}\right]\Upsilon\Psi_\perp\eta_0 > 0. \tag{2.27}$$

Then the ellipsoid $\mathcal{E}(P,\hat{x})$ lies in $\mathcal{F}_x$ and contains the solution set $\mathcal{X}$ if and only if there exists $\tau \geq 0$ such that

$$\left[\begin{array}{c|c} P & [I \;\; 0 \;\; \hat{x}]\Psi_\perp \\ \hline \Psi_\perp^T[I \;\; 0 \;\; \hat{x}]^T & \Psi_\perp^T\left(\mathrm{diag}\,(0,0,1) - \tau\Upsilon^T\left[\begin{array}{cc} I & 0 \\ 0 & -I \end{array}\right]\Upsilon\right)\Psi_\perp \end{array}\right] \succeq 0. \tag{2.28}$$

$\star$

The proof of this corollary follows immediately from the tightness of the embedding in Lemma A.5 in the unstructured case, and from the losslessness of the $\mathcal{S}$-procedure, under the assumption (2.27); see discussions following formulas (2.18) and (2.24). Minimizing the ellipsoid size $f(P)$ under constraint (2.28) then yields the globally optimal ellipsoid containing $\mathcal{X}$. We also notice that condition (2.27) is satisfied if and only if the kernel matrix has at least one (strictly) positive eigenvalue, and it is therefore easy to check.

2.1 Decoupled ellipsoid equations

We now build upon the LMI condition given in Theorem 2.1, in order to derive decoupled conditions for the optimal ellipsoid in terms of its shape matrix P and center $\hat{x}$ separately. These decoupled conditions are based on a variable elimination technique and permit to obtain explicit closed form results in the case of unstructured uncertainty. More fundamentally, they lead to a convex optimization problem of reduced numerical complexity with respect to the one given in Theorem 2.1, as it is discussed in detail in Section 2.2.

A first result is stated in the following corollary.

Corollary 2.2. *Let all symbols be defined as in Lemma 2.1, and define the partition*

$$Q(S,T,G) = \left[\begin{array}{c|c} Q_{11} & q_{12} \\ \hline q_{12}^T & 1-q_{22} \end{array}\right] \doteq \Psi_\perp^T\left(\mathrm{diag}\,(0,0,1) - \Omega(S,T,G)\right)\Psi_\perp, \tag{2.29}$$

$$B \doteq [I_n \;\; 0]\Psi_{\perp 1}. \tag{2.30}$$

Consider the optimization problem in the variables $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$

$$\text{minimize } f(BQ_{11}^\dagger B^T) \text{ subject to:} \tag{2.31}$$

$$S \succeq 0, T \succeq 0, \tag{2.32}$$

$$Q(S,T,G) \succeq 0, \tag{2.33}$$

$$(I - Q_{11}^\dagger Q_{11})B^T = 0. \tag{2.34}$$

If the above problem is feasible, then there exist a bounded ellipsoid that contains $\mathcal{X}$. In this case, calling $S_{opt}, T_{opt}, G_{opt}$ the optimal values of the problem variables, the ellipsoid $\mathcal{E}(P_{opt},\hat{x}_{opt})$ with

$$P_{opt} = BQ_{11}^\dagger(S_{opt},T_{opt},G_{opt})B^T \tag{2.35}$$

$$\hat{x}_{opt} = [I_n\, 0]\psi_{\perp 2} - BQ_{11}^\dagger(S_{opt},T_{opt},G_{opt})Q_{12} \tag{2.36}$$

is an outer ellipsoidal approximation of $\mathcal{X}$, that is optimal in the sense of the sufficient condition (2.19). This solution is equivalent to the one obtained minimizing $f(P)$ subject to the conditions in Theorem 2.1. $\star$

Proof. See Appendix B. □

Remark 2.2 (Boundedness). From Corollary 2.2 we immediately obtain a readily checkable sufficient condition for the solution set of uncertain linear equations to be bounded: If there exist $(S, T, G) \in \mathcal{B}(\mathbf{\Delta})$ such that (2.32)–(2.34) are satisfied, then the solution set $\mathcal{X}$ is bounded. These conditions become also necessary, under the hypotheses of Corollary 2.1.

Remark 2.3 (Emptiness and uniqueness). A preliminary analysis of (2.11) through (2.15) shows that a necessary condition in order to have (at least) one solution is that $y \in \mathcal{R}([A\ L])$. Notice also that if $\mathcal{N}([A\ L])$ is empty, then the uncertain linear equations may have at most one solution. In this case, the solution of the optimization problems in Theorem 2.1 and Corollary 2.2 would yield an ellipsoid reduced to a point, i.e. $P_{opt} = 0$.

Without need to solve any optimization problem, we may therefore conclude that:

$$\begin{aligned} \text{if } y \notin \mathcal{R}([A\ L]) \quad &\Rightarrow \quad \mathcal{X} \text{ is empty;} \\ \text{if } \mathcal{N}([A\ L]) = 0 \quad &\Rightarrow \quad \mathcal{X} \text{ is either empty or reduced to a point.} \end{aligned}$$

In the latter case, if $y \notin \mathcal{R}([A\ L]$ then $\mathcal{X}$ is certainly empty, otherwise the only candidate solution is of the form $\hat{x} = [I_n\ 0]\psi_{\perp 2}$, with $\psi_{\perp 2} \doteq [\hat{x}^T\ \hat{p}^T]^T$. To check if this is actually a solution, we can in some cases proceed by direct inspection. For instance, let $\hat{q} = R_A\hat{x} + H\hat{p} - R_y$, then in the case of unstructured uncertainty $\hat{x}$ is the unique solution if and only if $\hat{p}^T\hat{p} \leq \hat{q}^T\hat{q}$.

2.2 Analysis of numerical complexity

We next provide estimates of the numerical complexity of solving the ULE bounding problem, in both the coupled form of Theorem 2.1 and the decoupled form of Corollary 2.2. This analysis shows in particular that the formulation in Corollary 2.2 provides a drastic improvement in numerical efficiency with respect to the one in Theorem 2.1.

For the sake of clarity in the presentation, we here limit ourselves to the case of structured block-diagonal uncertainties of the type

$$\mathbf{\Delta} = \left\{ \Delta :\ \Delta = \mathrm{diag}\,(\Delta_1, \ldots, \Delta_\ell),\ \Delta_i \in \mathbb{R}^{d,d} \right\}$$

(all blocks of the same size), for which the scaling subspace is constituted of all triples S, T, G with $S = T = \mathrm{diag}\,(\lambda_1 I_d, \ldots, \lambda_\ell I_d)$, $G = 0$. Besides, within this section we shall assume the trace as the ellipsoid optimality criterion. Results similar to the ones reported below can also be determined if the log-determinant criterion is used instead of the trace.

Under the above hypotheses, the optimization problem (2.25), (2.19) derived from Theorem 2.1 is a convex semidefinite programming problem (SDP) involving a constraint matrix of dimension $M = n + (n - m + d\ell + 1) + \ell$,[1] and $N = n(n+1)/2 + n + \ell$ decision variables (the elements of $P, \hat{x}$ and $\lambda_1, \ldots, \lambda_\ell$). Therefore, using a general-purpose primal-dual interior-point SDP solver (i.e. we are in a sense considering a worst-case situation of a solver that does not exploit any possible structure in the problem) the complexity grows with problem size as

$$O(\sqrt{M})O(M^2N^2),$$

[1] We assumed Ψ full-rank, in order to fix the dimension of the orthogonal complement $\Psi_\perp$.

where the first factor denotes the number of outer iterations of the algorithm, and the second factor denotes the cost per iteration, see [39]. It is also observed in [39], Section 5.7 that in practice these algorithms behave much better that predicted by the above bound and that, in particular, the first factor can be assumed almost constant, so that the practical complexity is $O(M^2N^2)$. Notice however that in our context this gives $O(n^6+n^5d\ell+d^2(\ell^4+n^4\ell^2+n^2\ell^3))$, i.e. an $O(n^6)$ dependence on the dimension of x, and $O(d^2\ell^4)$ dependence on the size and number of uncertainty blocks.

Consider now the decoupled problem in Corollary 2.2, which is here rewritten in the equivalent form (all the previous hypotheses still holding)

$$\begin{aligned} &\inf \alpha \text{ subject to:} \\ &\quad \alpha - \operatorname{trace}\left(BQ_{11}^{-1}(\lambda_1,\ldots,\lambda_\ell)B^T\right) \geq 0 \\ &\quad \lambda_i \geq 0,\ i=1,\ldots,\ell \\ &\quad Q(\lambda_1,\ldots,\lambda_\ell) \succ 0, \end{aligned} \tag{2.37}$$

where $Q(\lambda_1,\ldots,\lambda_\ell)$ is affine in $\lambda_1,\ldots,\lambda_\ell$, see (2.29).

A basic idea for solving (2.37) is to associate a *barrier* for the feasible set, and solve a sequence of unconstrained minimization problems, involving a weighted combination of the barrier and the (linear) objective. The complexity of a path-following interior-point method as described in [25, p.93] depends on our ability of finding a 'self-concordant barrier' associated with the constraints. When such a barrier is known, the number of outer iterations grows as $O(\sqrt{\theta})$, where θ is the 'parameter of the barrier'. The cost of each iteration is proportional to that of computing the gradient g and Hessian $\mathcal{H}$ of the barrier, and solving the linear system $\mathcal{H}v = g$, where the unknown v is the search direction. We note again that in practice, the number of outer iterations is almost independent of problem size.

We can indeed find a self-concordant for problem (2.37), and find its parameter. To do this, we apply the addition rule [25, Prop. 5.1.3], which says that to find the barrier for multiple constraints, we simply add barriers and their respective parameters. The following is a direct consequence of the result [25, Prop. 5.1.8]: The function

$$F(\alpha,\lambda_1,\ldots,\lambda_\ell) = -\log\left(\alpha - \operatorname{trace}\left(BQ_{11}^{-1}(\lambda_1,\ldots,\lambda_\ell)B^T\right)\right) - \log\det Q(\lambda_1,\ldots,\lambda_\ell) - \sum_{i=1}^{\ell}\log\lambda_i \tag{2.38}$$

is a self-concordant barrier for problem (2.37), with parameter $\theta = 1+(\omega+1)+\ell = \omega+\ell+2$, where $\omega \doteq n-m+d\ell$ (note that B has size $n\times\omega$, and Q_{11} has size $\omega\times\omega$). A tedious but straightforward calculation shows that the cost of computing the gradient and Hessian of the barrier and solving for direction v is $O(\ell\omega^3+\ell^2\omega^2)$, hence the total complexity estimate is

$$O(\sqrt{\theta})O(\ell\omega^3+\ell^2\omega^2).$$

As noted above, the number of outer iterations is almost constant in practice, so the practical complexity can be assumed to be $O(\ell\omega^3+\ell^2\omega^2)$. From this, it results that the complexity of the decoupled problem is $O(n^3\ell+n^2d\ell^2+nd^2\ell^3+d^3\ell^4)$, which implies an $O(n^3)$ dependence on the dimension of x, and $O(d^3\ell^4)$ dependence on the size and number of uncertainty blocks. Hence, for fixed number and size of the uncertainty blocks, the decoupled problems improves upon the coupled one by a factor of $O(n^3)$.

2.3 Special case: unstructured uncertainty

In this section, we analyze in further detail the case when the uncertainty affecting the system of linear equations is *unstructured*, i.e. when Δ is a full matrix block.

For unstructured uncertainty the multipliers S, T, G simplify to $S = \lambda I$, $T = \lambda I$, $G = 0$. The matrices $Q_{11}(\lambda), q_{12}(\lambda), q_{22}(\lambda)$ defined in Corollary 2.2 are linear in λ, and it is convenient to express them as $Q_{11}(\lambda) = \lambda \bar{Q}_{11}$, $q_{12}(\lambda) = \lambda \bar{q}_{12}$, $q_{22}(\lambda) = \lambda \bar{q}_{22}$, with

$$\begin{aligned}
\bar{Q}_{11} &\doteq \Psi_{\perp 1}^T([0\ I]^T[0\ I] - [R_A\ H]^T[R_A\ H])\Psi_{\perp 1}, && (2.39)\\
\bar{q}_{12} &\doteq \Psi_{\perp 1}^T[R_A\ H]^T R_y + \Psi_{\perp 1}^T([0\ I]^T[0\ I] - [R_A\ H]^T[R_A\ H])\psi_{\perp 2}, && (2.40)\\
\bar{q}_{22} &\doteq R_y^T R_y - 2\psi_{\perp 2}^T[R_A\ H]^T R_y - \psi_{\perp 2}^T([0\ I]^T[0\ I] - [R_A\ H]^T[R_A\ H])\psi_{\perp 2}. && (2.41)
\end{aligned}$$

The optimal ellipsoid containing the solution set is in this case computable in closed form, as detailed in the following corollary.

Corollary 2.3. *Let $\mathbf{\Delta} = \mathbb{R}^{n_p, n_q}$, $B \doteq [I_n\ \ 0]\Psi_{\perp 1}$, and assume that $y \in \mathcal{R}([A\ \ L])$, (if this condition is not satisfied, the solution set is empty).*

Then, the solution set $\mathcal{X}$ is bounded if

$$\begin{aligned}
\bar{Q}_{11} &\succeq 0, && (2.42)\\
(I - \bar{Q}_{11}^\dagger \bar{Q}_{11})B &= 0, && (2.43)\\
(I - \bar{Q}_{11}^\dagger \bar{Q}_{11})\bar{q}_{12} &= 0. && (2.44)
\end{aligned}$$

The above conditions are also necessary, if there exists η_0 such that

$$\eta_0^T \begin{bmatrix} -\bar{Q}_{11} & \bar{q}_{12} \\ \bar{q}_{12}^T & \bar{q}_{22} \end{bmatrix} \eta_0 > 0. \qquad (2.45)$$

When (2.42)–(2.44) are satisfied, the optimal ellipsoid containing $\mathcal{X}$ is given by

$$\begin{aligned}
P_{opt} &= \frac{1}{\lambda_{opt}} B \bar{Q}_{11}^\dagger B^T && (2.46)\\
\hat{x}_{opt} &= [I_n\ \ 0]\psi_{\perp 2} - B\bar{Q}_{11}^\dagger \bar{q}_{12}, && (2.47)
\end{aligned}$$

with

$$\frac{1}{\lambda_{opt}} = \max\{\bar{q}_{22} + \bar{q}_{12}^T \bar{Q}_{11}^\dagger \bar{q}_{12}, 0\}.$$

When $P_{opt} = 0$ then the solution set contains at most one point. In particular, if $\bar{q}_{22} \geq 0$, then $\mathcal{X} = \{[I_n\ \ 0]\psi_{\perp 2}\}$, otherwise $\mathcal{X}$ is empty. $\star$

Proof. With the current scalings $S = \lambda I$, $T = \lambda I$, $G = 0$, problem (2.31)–(2.34) is easily restated as

$$\begin{aligned}
&\text{minimize } f(BQ_{11}^\dagger(\lambda)B^T) \text{ s.t.:}\\
&\quad \lambda \geq 0,\ Q_{11}(\lambda) \succeq 0,\\
&\quad 1 - q_{22}(\lambda) - q_{12}^T(\lambda) Q_{11}^\dagger(\lambda) q_{12}(\lambda) \geq 0,\\
&\quad (I - Q_{11}^\dagger(\lambda)Q_{11}(\lambda))q_{12}(\lambda) = 0,\\
&\quad (I - Q_{11}^\dagger(\lambda)Q_{11}(\lambda))B = 0.
\end{aligned}$$

Since all dependencies are linear in λ (and since $\lambda = 0$ cannot be optimal), the problem is equivalent to

$$\begin{aligned} \text{minimize } & f(\tfrac{1}{\lambda} B \bar{Q}_{11}^{\dagger} B^T) \text{ s.t.:} \\ & \lambda > 0,\ \bar{Q}_{11} \succeq 0, \\ & 1 - \lambda \bar{q}_{22} - \lambda \bar{q}_{12}^T Q_{11}^{\dagger} \bar{q}_{12} \geq 0, \\ & (I - \bar{Q}_{11}^{\dagger} \bar{Q}_{11}) \bar{q}_{12} = 0, \\ & (I - \bar{Q}_{11}^{\dagger} \bar{Q}_{11}) B = 0. \end{aligned}$$

Since $f(\cdot)$ is non-increasing in λ, if the problem is feasible the optimum is attained by the largest possible value of λ, i.e. for

$$\frac{1}{\lambda_{opt}} = \max\{\bar{q}_{22} + \bar{q}_{12}^T \bar{Q}_{11}^{\dagger} \bar{q}_{12}, 0\}.$$

The statements of the corollary then follow easily from the above considerations.

The necessity of conditions for boundedness of the solution set follows from the $\mathcal{S}$-procedure, see Lemma A.4. The last statement of the corollary follows from the discussion in Remark 2.3, noticing that $\hat{p}^T \hat{p} \leq \hat{q}^T \hat{q}$ if and only if $\bar{q}_{22} \geq 0$. □

Remark 2.4. Notice that, according to Lemma 2.1, in the unstructured case the actual solution set is the quadratic set

$$\mathcal{X} = \left\{ x = B\nu + [I_n\ 0]\psi_{\perp 2}, \nu : \begin{bmatrix} \nu \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{Q}_{11} & \bar{q}_{12} \\ \bar{q}_{12}^T & -\bar{q}_{22} \end{bmatrix} \begin{bmatrix} \nu \\ 1 \end{bmatrix} \leq 0 \right\}.$$

This set is indeed an ellipsoid, whenever $\bar{Q}_{11} \succ 0$ and $\bar{q}_{22} + \bar{q}_{12}^T \bar{Q}_{11}^{-1} \bar{q}_{12} > 0$, and hence Corollary 2.3 returns the exact solution set in this case. ◀

2.3.1 Additive unstructured uncertainty

An even more specialized case of the unstructured uncertainty situation above arises when the data A, y are affected by additive uncertainty, i.e.

$$[A(\Delta)\ y(\Delta)] = [A\ y] + L\Delta[R_A\ R_y],$$

with $L = \rho I_m$, $\rho > 0$, $[R_A\ R_y] = I_{n+1}$, $H = 0$, $\Delta \in \mathbb{R}^{m,n+1}$, and $\|\Delta\| \leq 1$.

In this case, we may choose the orthogonal complements as

$$\Psi_{\perp 1} = \begin{bmatrix} \rho I_n \\ -A \end{bmatrix};\ \psi_{\perp 2} = \begin{bmatrix} 0 \\ y/\rho \end{bmatrix},$$

and therefore $\bar{Q}_{11} = A^T A - \rho^2 I$, $\bar{q}_{12} = -A^T y/\rho$, $\bar{q}_{22} = 1 - y^T y/\rho^2$. The solution set is hence the quadratic set

$$\mathcal{X} = \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho^2} A^T A - I & -\frac{1}{\rho^2} A^T y \\ -\frac{1}{\rho^2} y^T A & \frac{1}{\rho^2} y^T y - 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0 \right\}. \tag{2.48}$$

In this simple situation, the set $\mathcal{X}$ could be analyzed directly, but we check what it is predicted by Corollary 2.3: Condition (2.45) is satisfied if and only if $\rho^2 > \lambda_{\min}\{[A\ y]^T [A\ y]\}$. In this case,

the solution set is bounded if and only if $\bar{Q}_{11} \succ 0$, i.e. for $\rho^2 < \lambda_{\min}\{A^T A\}$. On the other hand, if $\rho^2 < \lambda_{\min}\{[A\ y]^T[A\ y]\}$ then $\rho^2 < \lambda_{\min}\{A^T A\}$ and $\bar{q}_{22} < 0$,[2] therefore the solution set is empty. Lastly, we consider the situation when $\rho^2 = \lambda_{\min}\{[A\ y]^T[A\ y]\}$ and $\rho^2 < \lambda_{\min}\{A^T A\}$. In this case we have that the kernel matrix in (2.48) is positive semi-definite, hence the only points in $\mathcal{X}$ are those who annihilate the quadric

$$g(x) \doteq x^T(A^T A - \rho^2 I)x - 2y^T Ax + (y^T y - \rho^2).$$

Since $g(x)$ is strictly convex, it has the unique minimizer $\hat{x} = (A^T A - \rho^2 I)^{-1} A^T y$. Substituting $\hat{x}$ back into $g(x)$, we have that $\hat{x}$ is in the solution set if and only if $\rho^2(y^T(\rho^2 I - AA^T)^{-1}y - 1) = 0$; notice however that in the current situation, this latter condition is always satisfied.[3]

We may resume these results as follows.

- If $\lambda_{\min}\{[A\ y]^T[A\ y]\} < \rho^2 < \lambda_{\min}\{A^T A\}$, then $\mathcal{X}$ is an ellipsoid with

$$\begin{aligned} P_{opt} &= \alpha(A^T A - \rho^2 I)^{-1} \\ \hat{x}_{opt} &= (A^T A - \rho^2 I)^{-1} A^T y, \end{aligned}$$

  where $\alpha \doteq \rho^2(1 - y^T(\rho^2 I - AA^T)^{-1}y)$.

- If $\rho^2 < \lambda_{\min}\{[A\ y]^T[A\ y]\}$, then the solution set is empty.

- If $\rho^2 > \lambda_{\min}\{A^T A\}$, then the solution set is unbounded.

- If $\rho^2 = \lambda_{\min}\{[A\ y]^T[A\ y]\} < \lambda_{\min}\{A^T A\}$, then the solution set is the singleton $\mathcal{X} = \{\hat{x}_{opt}\}$.

It is worth to remark that $\rho^2 = \lambda_{\min}\{[A\ y]^T[A\ y]\}$ is the minimal perturbation size for which $\mathcal{X}$ is non-empty and that, in this case, the central solution $\hat{x}_{opt}$ coincides with the Total Least Squares solution (see for instance [18]) of the system of equations $([A\ \ y] + \Delta)\begin{bmatrix} x \\ -1 \end{bmatrix} = 0$.

The case when only the matrix A is uncertain (i.e. y is given and fixed) can be analyzed similarly, setting the LFR

$$[(A + \rho\Delta)\ \ y] = [A\ \ y] + L\Delta[R_A\ \ 0_n],$$

with $L = \rho I_m$, $R_A = I_n$. In this case we have $\bar{Q}_{11}, q_{12}$ as before, and $\bar{q}_{22} = -\frac{1}{\rho^2} y^T y$. The solution set is therefore the quadratic set

$$\mathcal{X} = \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho^2}A^T A - I & -\frac{1}{\rho^2}A^T y \\ -\frac{1}{\rho^2}y^T A & \frac{1}{\rho^2}y^T y \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0 \right\}. \tag{2.49}$$

---

[2] This is since $\lambda_{\min}\{[A\ y]^T[A\ y]\} \leq \lambda_{\min}\{A^T A\}$, and $\begin{bmatrix} -\bar{Q}_{11} & \bar{q}_{12} \\ \bar{q}_{12}^T & \bar{q}_{22} \end{bmatrix} \preceq 0$ if and only if $\rho^2 \leq \lambda_{\min}\{[A\ y]^T[A\ y]\}$.

[3] This fact can be proved as follows: $\rho^2 = \lambda_{\min}\{[A\ y]^T[A\ y]\}$ implies that $\det(\rho^2 I - [A\ y]^T[A\ y]) = 0$, but since $\rho^2 < \lambda_{\min}\{A^T A\}$, this is equivalent (by the Schur determinant rule) to $y^T(I - A(A^T A - \rho^2 I)^{-1}A^T)y - \rho^2 = 0$, which in turn, by application of matrix inversion lemma is equivalent to $\rho^2(y^T(\rho^2 I - AA^T)^{-1}y - 1) = 0$.

2.4 Numerical examples

2.4.1 Example 1

Consider the data

$$A(\Delta) = I_2 + 0.2\delta_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 0.5\delta_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \;\; y = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

with $|\delta_1| \leq 1$, $|\delta_2| \leq 1$. Here, the matrix $A(\Delta)$ is the identity, plus two additive perturbations. The uncertain data can be expressed in LFR format as

$$[A(\Delta) \mid y(\Delta)] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] + L\Delta[R_A \mid R_y], \tag{2.50}$$

$$L = \begin{bmatrix} 0.2 & 0 & 0 & 0.5 \\ 0 & -0.2 & -0.5 & 0 \end{bmatrix}, \;\; R_A = [I_2 \;\; I_2]^T, \;\; R_y = 0, \tag{2.51}$$

$\Delta = \mathrm{diag}\,(\delta_1 I_2, \delta_2 I_2)$, with $|\delta_1| \leq 1$, $|\delta_2| \leq 1$. The scaling subspace is in this case given by the set of triples (S, T, G) with $S = T = \mathrm{diag}\,(S_1, S_2)$, $S_1, S_2 \in \mathbb{R}^{2,2}$ symmetric, and $G = \mathrm{diag}\,(G_1, G_2)$, with $G_1, G_2 \in \mathbb{R}^{2,2}$ skew-symmetric.

To have an approximate idea of the shape of the solution set $\mathcal{X}$, we randomly generated a number of samples of δ_1, δ_2, and solved the corresponding linear equations. The points obtained are shown in Figure 1 (notice that the solution set of this ULE is *not* convex), together with the optimal bounding ellipsoid, determined by the solution of the convex problem in Theorem 2.1, having parameters

$$\hat{x} = \begin{bmatrix} 0.859 \\ 0.859 \end{bmatrix}; \;\; P = \begin{bmatrix} 0.462 & -0.246 \\ -0.246 & 0.462 \end{bmatrix}.$$

We next considered a modification of the previous example, in which $H \neq 0$. In particular, the ULE is now described by the LFR

$$[A(\Delta) \mid y(\Delta)] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] + L\Delta(I - H\Delta)^{-1}[R_A \mid R_y], \tag{2.52}$$

with

$$L = \begin{bmatrix} 0.2 & 0 & 0 & 0.5 \\ 0 & -0.2 & -0.5 & 0 \end{bmatrix}, \;\; R_A = [I_2 \;\; I_2]^T, \;\; R_y = 0, \;\; H = 0.5 I_4 \tag{2.53}$$

and $\Delta = \mathrm{diag}\,(\delta_1 I_2, \delta_2 I_2)$, $|\delta_1| \leq 1$, $|\delta_2| \leq 1$. The scaling subspace is the same as in the previous version of the example, and the application of Theorem 2.1 yields an optimal bounding ellipsoid with parameters

$$\hat{x} = \begin{bmatrix} 0.5687 \\ 1.0549 \end{bmatrix}; \;\; P = \begin{bmatrix} 0.8092 & -0.1759 \\ -0.1759 & 0.6578 \end{bmatrix}.$$

This optimal ellipsoid is depicted in Figure 2, together with randomly generated points in the interior of the solution set.

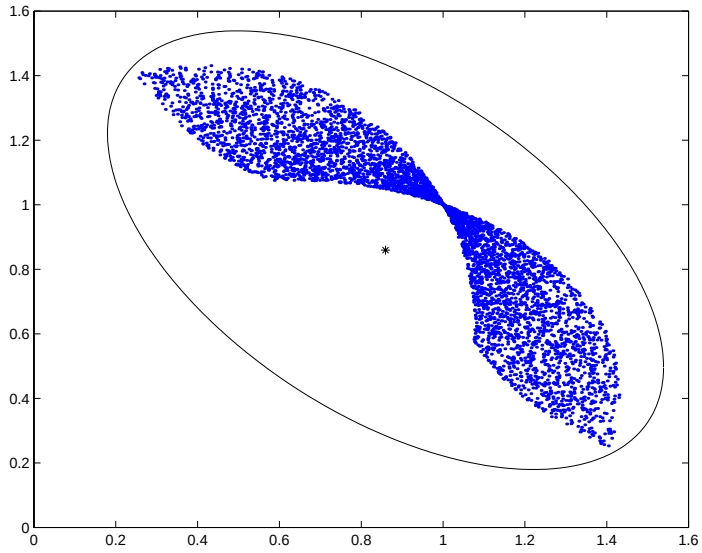

Figure 1: Solution set and bounding ellipsoid for the ULE resulting from the data in (2.50), (2.51) and structured uncertainty $\Delta = \text{diag}\,(\delta_1 I_2, \delta_2 I_2)$.

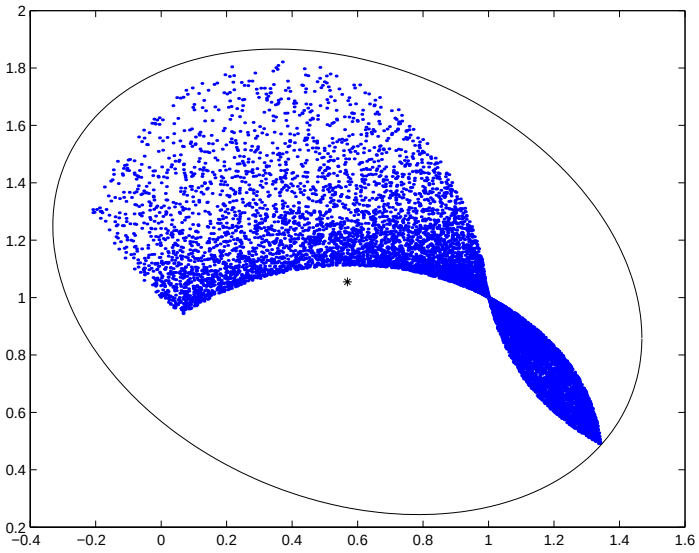

Figure 2: Solution set and bounding ellipsoid for the ULE resulting from the data in (2.52), (2.53) and structured uncertainty $\Delta = \text{diag}\,(\delta_1 I_2, \delta_2 I_2)$.

2.4.2 Example 2: FIR estimation

We next address the problem of estimating intervals of confidence for the coefficients of a discrete-time impulse response vector x, from uncertain input and output measurement. Specifically, the uncertain linear equation we consider has the form $A(\Delta)x = y(\Delta)$, where A is a lower-triangular Toeplitz matrix whose first column is the input vector u, and y is the system output. Both the

input u and output y are unknown-but-bounded componentwise, that is

$$|u_i - u_i^{nom}| \leq \rho, \quad i = 1, \ldots, n,$$
$$|y_i - y_i^{nom}| \leq \rho, \quad i = 1, \ldots, n,$$

where u^{nom} is the nominal input vector, y^{nom} is the measured output vector, and $\rho \geq 0$ is a given scalar. It is easy to derive a linear fractional representation for the uncertain system in this case. The uncertainty matrix has the following structure:

$$\Delta = \mathrm{diag}\,(\delta u_1 I_n, \delta u_2 I_{n-1}, \ldots, \delta u_n, \delta y_1, \delta y_2, \ldots, \delta y_n),$$

where δu_i (resp. δy_i) are the componentwise absolute errors in u (resp. y), normalized so that the uncertainty matrix is bounded in maximum singular value norm by one. The parameters of the linear fractional representation of the system are in this case

$$L = \rho \left[\begin{array}{c|c|c|c|c|c} I_n & \begin{bmatrix} 0_{1,n-1} \\ I_{n-1} \end{bmatrix} & \begin{bmatrix} 0_{2,n-2} \\ I_{n-2} \end{bmatrix} & \cdots & \begin{bmatrix} 0_{n-1,1} \\ 1 \end{bmatrix} & I_n \end{array}\right] \in \mathbb{R}^{n,n(n+1)/2+n},$$

$$R_A = \left[\begin{array}{c} I_n \\ \hline \begin{bmatrix} I_{n-1} & 0_{n-1,1} \\ I_{n-2} & 0_{n-2,2} \end{bmatrix} \\ \vdots \\ \hline \begin{bmatrix} 1 & 0_{1,n-1} \end{bmatrix} \\ \hline 0_{n,n} \end{array}\right] \in \mathbb{R}^{n(n+1)/2+n,n}, \quad R_y = \left[\begin{array}{c} 0_{n(n+1)/2,1} \\ \hline 1 \\ 1 \\ \vdots \\ 1 \end{array}\right] \in \mathbb{R}^{1,n(n+1)/2+n},$$

and $H = 0_{n(n+1)/2+n,n(n+1)/2+n}$. Taking $\rho = 0.1$, and $u^{nom} = \sin(i)$, vector y is generated by feeding this input to an FIR filter with coefficients $h^{nom} = \cos(i)$. For $n = 5$, the optimal result is

$$P = \begin{bmatrix} 0.1879 & -0.2669 & 0.1369 & -0.0700 & 0.1400 \\ -0.2669 & 0.6201 & -0.5059 & 0.2987 & -0.2961 \\ 0.1369 & -0.5059 & 1.0446 & -0.8017 & 0.6084 \\ -0.0700 & 0.2987 & -0.8017 & 1.4154 & -1.2740 \\ 0.1400 & -0.2961 & 0.6084 & -1.2740 & 2.4356 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} 0.6270 \\ -0.5851 \\ -0.8145 \\ -0.8395 \\ 0.5467 \end{bmatrix}.$$

This results in intervals of confidence for the coefficients of the FIR filter, obtained by projecting the ellipsoid of confidence on the coordinate axes:

$$\begin{bmatrix} 0.1935 \\ -1.3726 \\ -1.8365 \\ -2.0292 \\ -1.0140 \end{bmatrix} \leq x \leq \begin{bmatrix} 1.0606 \\ 0.2024 \\ 0.2076 \\ 0.3502 \\ 2.1073 \end{bmatrix}.$$

3 Set-valued prediction and filtering for uncertain systems

In this section, we study the problem of recursive ellipsoidal state bounding for uncertain discrete-time linear dynamical systems. First, we consider the set-valued prediction problem, i.e. given an ellipsoid $\mathcal{E}_k$ containing the state of the system at time k, we wish to determine an ellipsoid that contains the set of states that the system can achieve at time $k+1$, under a norm-bounded input and model uncertainty. Then, we discuss how this information can be recursively updated using uncertain measurement information. The objective of this section is to show that robust ellipsoidal prediction and filtering problems are just particular instances of the ULE problem discussed in the previous section and, as such, they are amenable to efficient numerical solution.

The approach taken here is derived from the deterministic interpretation of the discrete-time Kalman filter given in [3]. The deterministic filter in [3] was shown to give a state estimate in the form of an ellipsoidal set of all possible states consistent with the given measurements and a deterministic additive description of the noise. The idea of propagating ellipsoids of confidence for systems with ellipsoidal noise has been studied by several authors. Early contributions in this field are due to Schweppe [36], whose ideas were later developed by [5, 6, 20, 22, 29, 37], among many others. However, these authors consider the case with additive noise, assuming that the state-space process matrices are exactly known, in parallel to standard Kalman filtering.

The main contribution of this section is to extend the above mentioned set-valued approach to the model uncertainty case, i.e. when structured uncertainty affects *every* system matrix. Semidefinite relaxation techniques for this purpose have been first introduced by the authors in [8] for set-valued simulation, and in [12] for set-valued filtering. A particular case in which the system matrix uncertainty is jointly ellipsoidal-constrained with the process noises is also discussed in [32].

In the sequel, we derive general results for predictor/corrector filter recursions, using a unifying theoretical framework based on the uncertain linear equations paradigm discussed in the previous section. The proposed recursive filter has the classical predictor/corrector structure, therefore we first discuss ellipsoidal prediction, and then show how to update the predicted information with an upcoming measurement.

3.1 Prediction step

Consider the uncertain discrete-time linear system

$$x_{k+1} = \mathcal{A}_k(\tilde{\Delta}_k)x_k + \mathcal{B}_k(\tilde{\Delta}_k)u_k, \;\; k = 0, 1, \ldots \tag{3.54}$$

At a given time instant k, denote for ease of notation $x_k = x$, $u_k = u$, $\mathcal{A}_k = \mathcal{A}$, $\mathcal{B}_k = \mathcal{B}$, $\tilde{\Delta}_k = \tilde{\Delta}$, and $x_{k+1} = x_+$. Assume that $x \in \mathcal{E} = \mathcal{E}(P, \hat{x})$, $P = EE^T$, and $\|u\| \leq 1$, and let the system uncertainty be described in LFR form as

$$[\mathcal{A}(\tilde{\Delta}) \;\; \mathcal{B}(\tilde{\Delta})] = [\mathcal{A} \;\; \mathcal{B}] + \tilde{L}\tilde{\Delta}(I - \tilde{H}\tilde{\Delta})^{-1}[\tilde{R}_{\mathcal{A}} \;\; \tilde{R}_{\mathcal{B}}], \tag{3.55}$$

where $\mathcal{A} \in \mathbb{R}^{n,n}$, $\mathcal{B} \in \mathbb{R}^{n,n_u}$, $\tilde{L} \in \mathbb{R}^{n,n_p}$, $\tilde{R}_{\mathcal{A}} \in \mathbb{R}^{n_q,n}$, $\tilde{R}_{\mathcal{B}} \in \mathbb{R}^{n_q,n_u}$, $\tilde{H} \in \mathbb{R}^{n_q,n_p}$, and $\tilde{\Delta} \in \mathbf{\Delta}_1 \subset \mathbb{R}^{n_p,n_q}$, and let this LFR be well-posed over $\mathbf{\Delta}_1$. Let $\mathcal{X}_+$ denote the set of all one-step reachable states x_+, for $x \in \mathcal{E}$, $u : \|u\| \leq 1$, and $\tilde{\Delta} \in \mathbf{\Delta}_1$. Our objective is to determine a minimal ellipsoid $\mathcal{E}_+(P_+, \hat{x}_+)$ that is guaranteed to contain $\mathcal{X}_+$. To this end, a key observation is given in the following proposition.

Proposition 3.1. *The set $\mathcal{X}_+$ of states reachable (in one step) by system (3.54), for $x \in \mathcal{E}(P, \hat{x})$, $P = EE^T$, $u : \|u\| \leq 1$, and $\tilde{\Delta} \in \mathbf{\Delta}_1$, coincides with the solution set of the ULEs*

$$A(\Delta)x = y(\Delta),$$

with

$$[A(\Delta) \;\; y(\Delta)] \doteq [I_n \;\; \mathcal{A}\hat{x}] + L\Delta(I - H\Delta)^{-1}[0_{n_q+2,n} \;\; R_y], \tag{3.56}$$

where

$$\begin{aligned} L &= \begin{bmatrix} \tilde{L} & \mathcal{A}E & \mathcal{B} \end{bmatrix} \\ H &= \begin{bmatrix} \tilde{H} & \tilde{R}_{\mathcal{A}}E & \tilde{R}_{\mathcal{B}} \\ 0_{2,n_p} & 0_{2,n} & 0_{2,n_u} \end{bmatrix} \\ R_y &= \begin{bmatrix} \tilde{R}_{\mathcal{A}}\hat{x} \\ 1 \\ 1 \end{bmatrix}, \end{aligned}$$

and $\Delta = \mathrm{diag}\,(\tilde{\Delta}, \delta_x, \delta_u)$, $\delta_x \in \mathbb{R}^n$, $\delta_u \in \mathbb{R}^{n_u}$, and $\|\Delta\| \leq 1$. ⋆

Proof. Observe first that the set of $x \in \mathcal{E}$ and $u : \|u\| \leq 1$ can be described in LFR format as

$$\begin{bmatrix} x(\delta_x) \\ u(\delta_u) \end{bmatrix} = \begin{bmatrix} \hat{x} \\ 0 \end{bmatrix} + \mathrm{diag}\,(E, I_{n_u})\mathrm{diag}\,(\delta_x, \delta_u) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $\delta_x \in \mathbb{R}^n$, $\delta_u \in \mathbb{R}^{n_u}$, and $\|\delta_x\|, \|\delta_u\| \leq 1$. Hence, it is clear from (3.54) that the reachable states coincide with the solution set of the uncertain linear equations $I_n x_+ = y(\Delta)$, where $y(\Delta)$ is the product of LFRs

$$y(\Delta) \doteq [\mathcal{A}_k(\tilde{\Delta}_k) \;\; \mathcal{B}_k(\tilde{\Delta}_k)] \begin{bmatrix} x(\delta_x) \\ u(\delta_u) \end{bmatrix},$$

$\Delta = \mathrm{diag}\,(\tilde{\Delta}, \delta_x, \delta_u)$. The LFR representation of $y(\Delta)$ is constructed by using a standard rule for multiplication of LFRs (see for instance [41]), from which the proposition statement immediately follows. □

An immediate consequence of the above proposition is that a sub-optimal bounding ellipsoid for the reachable set $\mathcal{X}_+$ can be obtained applying Theorem 2.1 or Corollary 2.2 to the ULEs in (3.56).

Remark 3.1. If we specialize the ULEs above to the case when no model uncertainty is present, but only additive noise is acting on the system, then straightforward (but tedious) manipulations show that, if the trace criterion is chosen, this specific instance of the optimization problem in Corollary 2.2 can be solved in closed form and yields an optimal ellipsoid with center in $\hat{x}_+ = \mathcal{A}\hat{x}$ and shape matrix

$$P_+ = \frac{1}{\tau_x}\mathcal{A}P\mathcal{A}^T + \frac{1}{\tau_u}\mathcal{B}\mathcal{B}^T,$$

with

$$\tau_x = \frac{\sqrt{\mathrm{trace}\,\mathcal{A}P\mathcal{A}^T}}{\sqrt{\mathrm{trace}\,\mathcal{A}P\mathcal{A}^T} + \sqrt{\mathrm{trace}\,\mathcal{B}\mathcal{B}^T}}, \quad \tau_u = \frac{\sqrt{\mathrm{trace}\,\mathcal{B}\mathcal{B}^T}}{\sqrt{\mathrm{trace}\,\mathcal{A}P\mathcal{A}^T} + \sqrt{\mathrm{trace}\,\mathcal{B}\mathcal{B}^T}}.$$

It can also be observed that in this case the semidefinite relaxation scheme searches the optimum over a parameterized family of ellipsoids that coincides with the classical Schweppe ellipsoidal family, [35]. The same parameterized family is used in [6] (Theorem 4.2) for approximating the sum of k given ellipsoids.

We also remark that, in the considered particular case, a recent result of Ben-Tal and Nemirovski [1] provides an assessment on the tightness of the approximation. In particular, they proved that the ratio between the volume of the volume-optimal ellipsoid obtained by the semidefinite relaxation method and the volume of the best possible bounding ellipsoid is less than $(\pi/2)^{n/2}$, where n is the dimension of the state vector x. Using a technique similar to that of [1, 26], it can be proved that if $\mathcal{E}^*$ is the minimum trace ellipsoid obtained via the semidefinite relaxation method and $\mathcal{E}_*$ is the minimum trace ellipsoid among all possible ellipsoids, then

$$\frac{\sqrt{\operatorname{trace} P_+^*}}{\sqrt{\operatorname{trace} P_{+*}}} \leq \sqrt{\frac{\pi}{2}} \simeq 1.253.$$

Notice further that the above bound states that the size of the sub-optimal bounding ellipsoid is at most 25.3% larger that the actual optimal size, and that this figure holds independently of the state dimension. The interested reader is also addressed to [16, 21, 24] for further discussion of semidefinite relaxations of non-convex quadratic problems. ◀

3.2 Measurement update step

Suppose that at time instant $k+1$ we are given the information coming from the prediction step:

$$x_+ \in \mathcal{E}_+(P_+, \hat{x}_+). \tag{3.57}$$

Now, an observation (measurement) z_+ of the state x_+ becomes available, where

$$z_+ = \mathcal{C}(\tilde{\Delta})x_+ + \mathcal{D}(\tilde{\Delta})v, \tag{3.58}$$

with

$$[\mathcal{C}(\tilde{\Delta})\ \ \mathcal{D}(\tilde{\Delta})] = [\mathcal{C}\ \ \mathcal{D}] + \tilde{L}\tilde{\Delta}(I - \tilde{H}\tilde{\Delta})^{-1}[\tilde{R}_{\mathcal{C}}\ \ \tilde{R}_{\mathcal{D}}],$$

where $\mathcal{C} \in \mathbb{R}^{n_z,n}$, $\mathcal{D} \in \mathbb{R}^{n_z,n_v}$, $\tilde{L} \in \mathbb{R}^{n_z,n_p}$, $\tilde{R}_{\mathcal{C}} \in \mathbb{R}^{n_q,n}$, $\tilde{R}_{\mathcal{D}} \in \mathbb{R}^{n_q,n_v}$, $\tilde{H} \in \mathbb{R}^{n_q,n_p}$, $\tilde{\Delta} \in \mathbf{\Delta}_1 \subset \mathbb{R}^{n_p,n_q}$, $v \in \mathbb{R}^{n_v}$, $\|v\| \leq 1$ is a measurement noise term, and we assume that the above LFR is well-posed over $\mathbf{\Delta}_1$.

As in a standard filtering problem, we want to update the predicted state estimate $\mathcal{E}_+$ with the information carried by the measurement, and determine a minimal *filtered* ellipsoid $\mathcal{E}_{+|+}(P_{+|+}, \hat{x}_{+|+})$ which contains the states that are consistent with both the prediction *and* the measurement.

The reasoning is similar to the previous one: Let $P_+ = E_+E_+^T$, then it is known from the prediction step that $x_+ = \hat{x}_+ + E_+\delta_x$, for some $\|\delta_x\| \leq 1$. Considering also the measurement equation (3.58), we see that the admissible states x_+ (i.e. those which are consistent with both the prediction and measurement) are the solutions of the uncertain linear equations

$$\begin{aligned} I_n x_+ &= \hat{x}_+ + E_+\delta_x \\ \mathcal{C}(\tilde{\Delta})x_+ &= z_+ - \mathcal{D}(\tilde{\Delta})v. \end{aligned}$$

Applying standard rules of LFR algebra, we can construct an explicit linear fractional representation for these equations, and hence apply the results in Theorem 2.1 or Corollary 2.2 to numerically compute the filtered ellipsoid. The following proposition explicitly reports the representation for the ULEs to be used in the measurement update step.

Proposition 3.2. *The set $\mathcal{X}_{+|+}$ of states consistent with set-valued prediction (3.57) and measurement model (3.58), coincides with the solution set of the ULEs*

$$A(\Delta)x = y(\Delta),$$

with

$$[A(\Delta) \;\; y(\Delta)] \doteq \begin{bmatrix} \mathcal{C} & z_+ \\ I_n & \hat{x}_+ \end{bmatrix} + L\Delta(I - H\Delta)^{-1}[R_A \;\; R_y], \tag{3.59}$$

where

$$\begin{aligned} L &= \begin{bmatrix} \tilde{L} & -\mathcal{D} & 0_{n_z,n} \\ 0_{n,n_p} & 0_{n,n_v} & E_+ \end{bmatrix} \\ H &= \begin{bmatrix} \tilde{H} & -\tilde{R}_{\mathcal{D}} & 0_{n_q,n} \\ 0_{2,n_p} & 0_{2,n_v} & 0_{2,n} \end{bmatrix} \\ R_A &= \begin{bmatrix} \tilde{R}_{\mathcal{C}} \\ 0_{2,n} \end{bmatrix}; \quad R_y = \begin{bmatrix} 0_{n_q,1} \\ 1_{2,1} \end{bmatrix}, \end{aligned}$$

$\Delta = \mathrm{diag}\,(\tilde{\Delta}, \delta_v, \delta_x)$, $\delta_x \in \mathbb{R}^n$, $\delta_v \in \mathbb{R}^{n_v}$, *and* $\|\Delta\| \leq 1$. ⋆

Remark 3.2. In the case when there is no model uncertainty, but only additive noise v is present in the measurement equation, then the update problem reduces to the classical one of bounding the intersection of two ellipsoids, see for instance [6], Section 5. Observe that in this case $\Delta = \mathrm{diag}\,(\delta_v, \delta_x)$, and the scalings are $S = \mathrm{diag}\,(\lambda_v I_{n_u}, \lambda_x I_n)$, $T = \mathrm{diag}\,(\lambda_v, \lambda_x)$, $G = 0$. Therefore, the convex optimization problem in Corollary 2.2 involves only the two scalar variables λ_v, λ_x. ◀

3.3 Numerical example

In the following example, we illustrate the mechanism of the robust ellipsoidal bounding algorithms for one-step-ahead prediction, and the successive measurement update.

Consider first the setup of Section 3.1, with

$$\begin{aligned} &[\mathcal{A}(\tilde{\Delta}) \;\; \mathcal{B}(\tilde{\Delta})] = \\ &\left[\begin{array}{cc|c} 0.2 & 1 & 0 \\ -0.5 & 0.3 & 0.01 \end{array}\right] + \tilde{L}\tilde{\Delta}[\tilde{R}_{\mathcal{A}} \mid \tilde{R}_{\mathcal{B}}] = \left[\begin{array}{cc|c} 0.2 & 1 & 0 \\ -0.5 + 0.4\delta_1 & 0.3 & 0.03 \end{array}\right], \end{aligned}$$

$$\tilde{L} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \;\; \tilde{R}_{\mathcal{A}} = [1 \;\; 0], \;\; \tilde{R}_{\mathcal{B}} = 0,$$

$\tilde{\Delta} = \delta_1$, and $|\delta_1| \leq 1$. Let x_k belong to the ellipsoid $\mathcal{E}$ having center $\hat{x} = [0 \;\; 0]^T$ and shape matrix $P = \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix}$. Applying Proposition 3.1 and successively Corollary 2.2 (using the trace

criterion) on the resulting ULEs, we determined the optimal bounding ellipsoid $\mathcal{E}_+$ for the states at time $k+1$, having center $\hat{x}_+ = [0\ 0]^T$ and $P_+ = \begin{bmatrix} 0.0008 & -0.0021 \\ -0.0021 & 0.0121 \end{bmatrix}$. In order to give a pictorial idea of the reachable set at time $k+1$, Figure 3 shows the predicted ellipsoid $\mathcal{E}_+$, together with random points in $\mathcal{X}_+$.

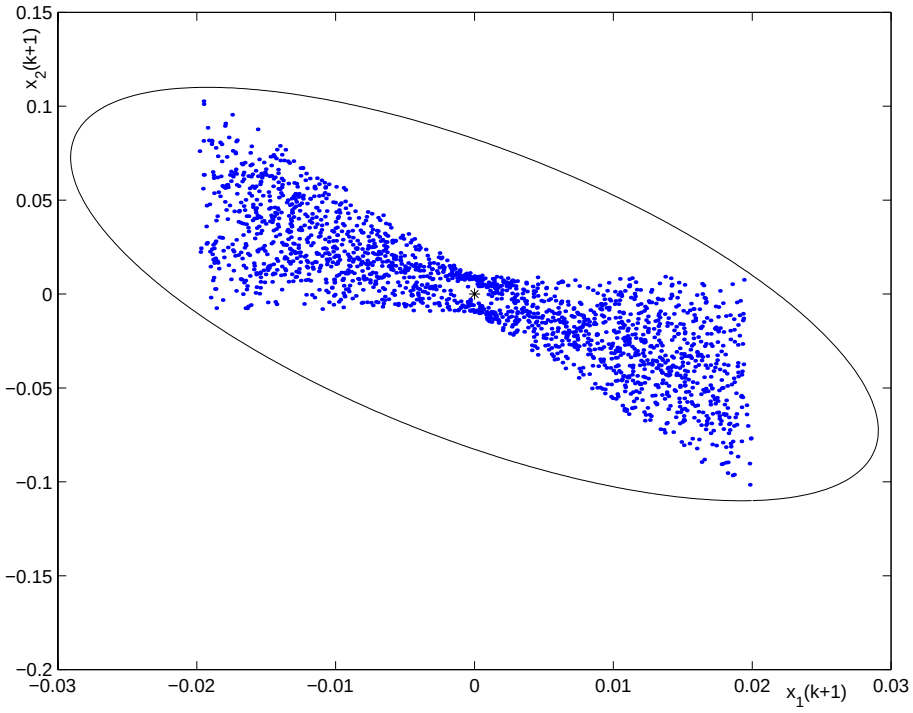

Figure 3: Optimal predicted ellipsoid $\mathcal{E}_+$ and random points in $\mathcal{X}_+$.

Consider now the setup of Section 3.2, and assume that the measurement $z_+ = 0.018$ becomes available, according to the model (3.58), with

$$[\tilde{C}(\tilde{\Delta})\ \ \tilde{D}(\tilde{\Delta})] = \left[\begin{array}{cc|c} 1 & 0 & 0.005 \end{array}\right] + \tilde{L}\tilde{\Delta}[\tilde{R}_{\mathcal{C}} \mid \tilde{R}_{\mathcal{D}}] = \left[\begin{array}{cc|c} 1+0.01\delta_2 & 0 & 0.005 \end{array}\right],$$

$\tilde{\Delta} = \delta_2$, $|\delta_2| \leq 1$, and $\tilde{L} = 0.01, \tilde{R}_{\mathcal{C}} = [1\ 0], \tilde{R}_{\mathcal{D}} = 0$. Then, we update the prediction information with this new measurement, constructing the ULEs according to Proposition 3.2 and solving the relative optimization problem in Corollary 2.2 (using the trace criterion). The updated ellipsoid $\mathcal{E}_{+|+}$ has center in $\hat{x}_{+|+} = [0.0149\ \ -0.0373]^T$ and shape matrix $P_{+|+} = \begin{bmatrix} 0.0001 & -0.0003 \\ -0.0003 & 0.0064 \end{bmatrix}$. Figure 4 shows the shrunk updated ellipsoid in comparison with the predicted one. In a dynamic setting, the process is then iteratively repeated predicting forward in time $\mathcal{E}_{+|+}$ through the uncertain dynamic equations (3.54), etc...

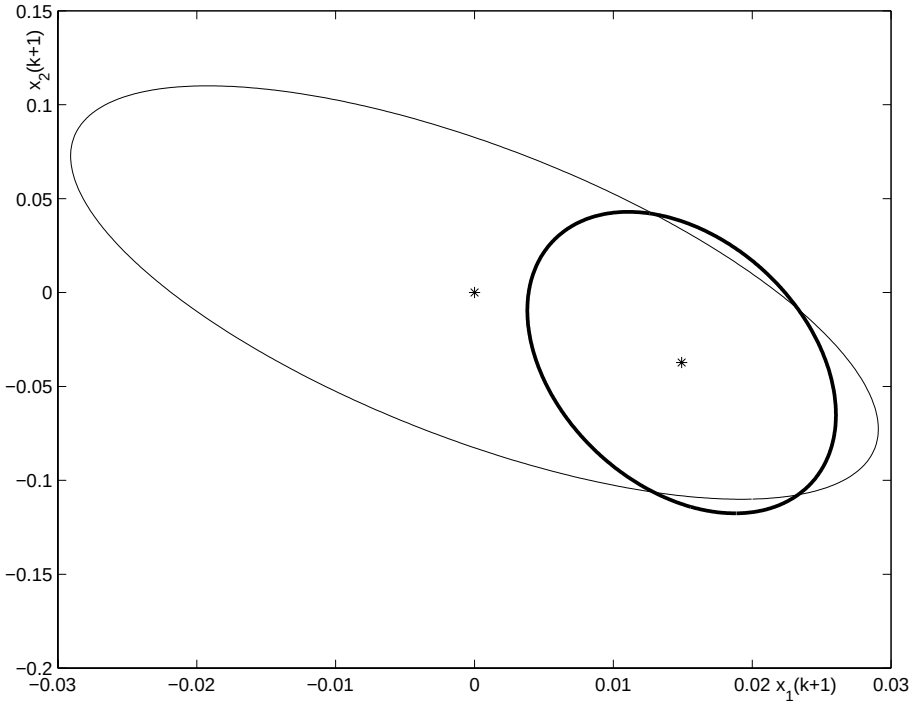

Figure 4: Predicted ellipsoid $\mathcal{E}_+$ (light line) and updated ellipsoid $\mathcal{E}_{+|+}$ (bold line).

4 Conclusions

In this paper, we presented a comprehensive discussion of semidefinite relaxation methods for approximation problems that arise in the context of systems affected by unknown-but-bounded uncertainties. A main points of the paper is to introduce the uncertain linear equations paradigm and to show that bounds on the solution set can be obtained efficiently via convex optimization. Besides being of interest in its own right from a theoretical point of view, the ULE model has direct application in many engineering problems, and in particular in system identification and set-membership prediction and filtering.

In this latter respect, the results in Section 3 extend the existing literature on set-membership filtering to the case when uncertainty is present in the system description. When there is no uncertainty in the system description, but only unknown-but-bounded additive noise is present, we recover classical ellipsoidal filtering results.

Some comments are in order regarding the employed methodology. We remark that all the discussed problems are numerically hard to solve exactly. From a practical point of view, the standard semidefinite relaxation that we use provides a viable methodology for computing a sub-optimal solution, at a provably low computational cost (basically $O(n^3)$, see Section 2.2). In some special cases, we pointed out that the approximation is actually exact (Corollary 2.1), or a precise bound on the conservatism can be assessed, see Remark 3.1. However, no result on the sharpness of the approximation is to date available for the general case, and this issue is currently a hot research topic, see e.g. [2, 14, 21, 26]. In principle, the sharpness of the approximation can be improved using higher order semidefinite relaxations, at the expense of increased computational burden, see

[30], Chapter 6, and [21] for further results along this line.

Acknowledgments

We gratefully acknowledge Arkadi Nemirovski for a precious suggestion regarding the barrier function used in Section 2.2.

Appendix

A LMI technical lemmas

For easier reference of the reader, this section gathers several technical results on LMI manipulation. Most of these results can be found in standard texts such as [4, 38].

Lemma A.1 (Projection). *Let $F = F^T$. The inequality $\xi^T F \xi \leq 0$ holds for all ξ: $Q\xi = 0$, if and only if $Q_\perp^T F Q_\perp \preceq 0$.*

Lemma A.2 (Well-posedness). *The LFR $M(\Delta) = M + L\Delta(I - H\Delta)^{-1}R$ is well-posed over $\mathbf{\Delta}_1$ if and only if $\det(I - H\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}_1$. A sufficient condition for well-posedness is: there exist a triple $(S, T, G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succ 0$, $T \succ 0$ such that*

$$\begin{bmatrix} H \\ I \end{bmatrix}^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \begin{bmatrix} H \\ I \end{bmatrix} \preceq 0.$$

The above condition is also necessary in the unstructured case, i.e. when $\mathbf{\Delta} = \mathbb{R}^{n_p, n_q}$.

This lemma was first derived in the context of μ-analysis in [9]. A proof of the results in the form given here may be found in [13].

Lemma A.3 (Homogenization). *Let $T = T^T$. The following two conditions are equivalent.*

$$\text{(a)} \quad \begin{bmatrix} \xi \\ 1 \end{bmatrix}^T \begin{bmatrix} T & u \\ u^T & v \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \geq 0 \text{ for all } \xi;$$

$$\text{(b)} \quad \begin{bmatrix} T & u \\ u^T & v \end{bmatrix} \succeq 0.$$

Proof. *The implication from (b) to (a) is trivial. We show that (a) implies (b) by contradiction. Suppose $\exists\, \bar{\xi}, \alpha$ such that $\begin{bmatrix} \bar{\xi}^T & \alpha \end{bmatrix} \begin{bmatrix} T & u \\ u^T & v \end{bmatrix} \begin{bmatrix} \bar{\xi}^T & \alpha \end{bmatrix}^T < 0$. Then, if $\alpha \neq 0$, dividing both sides by α^2, we get $\begin{bmatrix} \frac{1}{\alpha}\bar{\xi}^T & 1 \end{bmatrix} \begin{bmatrix} T & u \\ u^T & v \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha}\bar{\xi}^T & 1 \end{bmatrix}^T < 0$, which clearly contradicts the hypothesis (a). On the other hand, $\alpha = 0$ would imply that $\bar{\xi}^T T \bar{\xi} < 0$. Choosing then $\xi = \beta\bar{\xi}$ and substituting in (a) we have*

$$\beta^2(\bar{\xi}^T T \bar{\xi}) + 2\beta u^T \bar{\xi} + v, \tag{A.60}$$

which is a concave parabola in β, since $\bar{\xi}^T T \bar{\xi} < 0$. Therefore, there will exist a value of β such that (A.60) is negative, which contradicts the hypothesis. □

Lemma A.4 ($\mathcal{S}$-procedure). *Let $F_0(\xi), F_1(\xi), \ldots, F_p(\xi)$ be quadratic functions in the variable $\xi \in \mathbb{R}^n$*

$$F_i(\xi) = \xi^T T_i \xi + 2u_i^T \xi + v_i,\ i = 0, \ldots, p,$$

with $T_i = T_i^T$. Then, the implication

$$F_1(\xi) \geq 0, \ldots, F_p(\xi) \geq 0 \ \Rightarrow\ F_0(\xi) \geq 0 \tag{A.61}$$

holds if there exist $\tau_1, \ldots, \tau_p \geq 0$ such that

$$F_0(\xi) - \sum_{i=1}^{p} \tau_i F_i(\xi) \geq 0,\ \forall \xi. \tag{A.62}$$

When $p = 1$, condition (A.62) is also necessary for (A.61), provided there exist some ξ_0 such that $F_1(\xi_0) > 0$. An extension of this result to the case of $p = 2$ can be found in [31]. Notice also that, by homogenization, condition (A.62) is equivalent to

$$\exists \tau_1, \ldots, \tau_p \geq 0 \text{ such that } \begin{bmatrix} T_0 & u_0 \\ u_0^T & v_0 \end{bmatrix} - \sum_{i=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix} \succeq 0. \tag{A.63}$$

Lemma A.5 (Quadratic embedding). *Let $\mathcal{Q} \doteq \left\{ \begin{bmatrix} q^T & p^T \end{bmatrix}^T : \ p = \Delta q \text{ for some } \Delta \in \mathbf{\Delta}_1 \right\}$, and $\mathcal{B}(\mathbf{\Delta}) = \left\{ (S,T,G) \ : \ \forall \Delta \in \mathbf{\Delta},\ S\Delta = \Delta T,\ G\Delta = -\Delta^T G^T \right\}$. For any triple $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$, define the set*

$$\mathcal{Q}_{S,T,G} \doteq \left\{ \begin{bmatrix} q \\ p \end{bmatrix} : \ \begin{bmatrix} q \\ p \end{bmatrix}^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \geq 0 \right\}. \tag{A.64}$$

Then $\mathcal{Q} \subseteq \mathcal{Q}_{S,T,G}$.

In the case of unstructured uncertainty $\mathbf{\Delta} \equiv \mathbb{R}^{n_p, n_q}$ we have in particular that $\mathcal{Q} \equiv \mathcal{Q}_{S,T,G}$, for $S = \lambda I_{n_p}$, $T = \lambda I_{n_q}$, $\lambda \in \mathbb{R}$, and $G = 0$, $\lambda > 0$.

Proof. Let $[q^T \ p^T]^T \in \mathcal{Q}$, then for any $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$ we have that $q^T G p = q^T G \Delta q = 0$, by the skew-symmetry of $G\Delta$. In addition, we have

$$\begin{aligned} q^T T q - p^T S p &= q^T (T - \Delta^T S \Delta) q \\ &= q^T (T - \Delta^T \Delta T) q \succeq 0. \end{aligned}$$

In the above, we have used the fact that, since $S\Delta = \Delta T$, the matrix $\Delta^T \Delta T$ is symmetric, then T commutes in the product with $\Delta^T \Delta$, and therefore these two matrices are simultaneously diagonalizable ([17], Corollary 4.5.18), i.e. we may write the factorizations $T = V J_T V^T$, $\Delta^T \Delta = V J_\Delta V^T$, where J_T, J_Δ are diagonal, and V is orthogonal. It then follows that the eigenvalues of $T - \Delta^T \Delta T$ are the diagonal terms of $(I - J_\Delta) J_T$, which are non-negative, if $T \succeq 0$. The previous conditions are written compactly as

$$\begin{bmatrix} q \\ p \end{bmatrix}^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \geq 0, \tag{A.65}$$

which proves the first part of the lemma.

To prove the second part of the lemma, we consider the case of unstructured uncertainty, i.e. $\mathbf{\Delta} = \mathbb{R}^{n_p,n_q}$ (only one full block). In this case the set $\mathcal{B}(\mathbf{\Delta})$ reduces to the set of triples (S,T,G), with $S = \lambda I_{n_p}$, $T = \lambda I_{n_q}$, $\lambda \in \mathbb{R}$, and $G = 0$. Clearly, $p = \Delta q$ for some $\Delta : \ \|\Delta\| \leq 1$ if and only if $p^T p \leq q^T q$, which is equivalent to (A.65), for any $\lambda > 0$. □

Lemma A.6 (Schur complements). *The condition*

$$\left[\begin{array}{cc} A & B \\ B^T & D \end{array} \right] \succeq 0$$

is equivalent to

$$D \succeq 0,\ A - BD^\dagger B^T \succeq 0,\ (I - D^\dagger D)B^T = 0$$

and also to

$$A \succeq 0,\ D - B^T A^\dagger B \succeq 0,\ (I - A^\dagger A)B = 0,$$

where $A^\dagger$, $D^\dagger$ denote the Moore-Penrose pseudoinverse of A and D, respectively. Notice that the condition $(I - A^\dagger A)B = 0$ means that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Similarly, the condition $(I - D^\dagger D)B^T = 0$ means that $\mathcal{N}(D) \subseteq \mathcal{N}(B)$ or, equivalently, that $\mathcal{R}(B^T) \subseteq \mathcal{R}(D)$.

Lemma A.7 (Block elimination). *Let $Q_{11} = Q_{11}^T$, $Q_{22} = Q_{22}^T$. There exist matrices $X = X^T$ and Z such that*

$$\left[\begin{array}{ccc} X & Z & B \\ Z^T & Q_{11} & Q_{12} \\ B^T & Q_{12}^T & Q_{22} \end{array} \right] \succeq 0 \tag{A.66}$$

if and only if

$$\left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{array} \right] \ \succeq \ 0, \ \textit{and} \tag{A.67}$$

$$\exists X = X^T : \left[\begin{array}{cc} X & B \\ B^T & Q_{22} \end{array} \right] \ \succeq \ 0. \tag{A.68}$$

Proof. This result is closely related to the variable elimination lemma well-known in LMI literature (see for instance [38], Theorem 2.3.12). We next report a proof for our specific formulation of the lemma.

The implication from (A.66) to (A.67)–(A.68) is straightforward, since if a matrix is positive semi-definite, so are all principal sub-matrices. For the converse, notice that, by Lemma A.6, condition (A.66) is equivalent to

$$Q_{22} \ \succeq \ 0 \tag{A.69}$$

$$\left[\begin{array}{cc} X & Z \\ Z^T & Q_{11} \end{array} \right] - \left[\begin{array}{c} B \\ Q_{12} \end{array} \right] Q_{22}^\dagger \left[\begin{array}{c} B \\ Q_{12} \end{array} \right]^T \ \succeq \ 0 \tag{A.70}$$

$$(I - Q_{22}^\dagger Q_{22}) \left[\begin{array}{c} B \\ Q_{12} \end{array} \right]^T \ = \ 0. \tag{A.71}$$

Clearly, (A.67) implies $(I - Q_{22}^{\dagger}Q_{22})Q_{12}^T = 0$, and (A.68) implies $(I - Q_{22}^{\dagger}Q_{22})B^T = 0$, therefore (A.67)–(A.68) imply (A.71). Define now

$$\begin{aligned} \bar{X} &\doteq BQ_{22}^{\dagger}B^T && \text{(A.72)} \\ \bar{Z} &\doteq BQ_{22}^{\dagger}Q_{12}^T && \text{(A.73)} \\ \bar{Q}_{11} &\doteq Q_{11} - Q_{12}Q_{22}^{\dagger}Q_{12}^T, && \text{(A.74)} \end{aligned}$$

then (A.70) writes

$$\left[\begin{array}{cc} X - \bar{X} & (Z - \bar{Z}) \\ (Z - \bar{Z})^T & \bar{Q}_{11} \end{array} \right] \succeq 0. \tag{A.75}$$

From (A.67) it follows that $\bar{Q}_{11} \succeq 0$, therefore (A.75) is feasible for $X = \bar{X}$, $Z = \bar{Z}$, which concludes the proof. □

Corollary A.1 (Decoupling). *Let all symbols be defined as in Lemma A.7, and let*

$$\left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{array} \right] \succeq 0.$$

Then the problem

$$\min_{X,Z} f(X) \text{ subject to } (A.66) \tag{A.76}$$

is equivalent to

$$\min_{X} f(X) \text{ subject to } (A.68). \tag{A.77}$$

Moreover, if problem (A.77) is feasible and $f(\cdot)$ is either the trace function $f(X) = \mathrm{trace}\,(X)$, or the log-det function $f(X) = \log\det(X)$, then problem (A.76) has a unique optimal solution given by

$$\begin{aligned} \bar{X} &\doteq BQ_{22}^{\dagger}B^T && \text{(A.78)} \\ \bar{Z} &\doteq BQ_{22}^{\dagger}Q_{12}^T && \text{(A.79)} \end{aligned}$$

Proof. When (A.67) holds, we know from Lemma A.7 that (A.66) is feasible if and only if (A.68) is feasible, which immediately proves the equivalence between problems (A.76) and (A.77).

If the latter is feasible, then (A.66) is also feasible, and therefore (A.75) holds (with the symbols defined in (A.72)–(A.74)), which means that

$$\begin{aligned} &X \succeq \bar{X} + (Z - \bar{Z})\bar{Q}_{11}^{\dagger}(Z - \bar{Z})^T \\ &(I - \bar{Q}_{11}^{\dagger}\bar{Q}_{11})(Z - \bar{Z})^T = 0. \end{aligned}$$

Now, since $f(X_1) \geq f(X_2)$ whenever $X_1 \succeq X_2$ (both in the case of trace and log-determinant) the minimum of $f(X)$ is achieved for $X = \bar{X}$, $Z = \bar{Z}$. □

B Proof of Corollary 2.2

With the position (2.15), let us define the following partitions

$$\begin{aligned}\left[\begin{array}{c|c} [I_n \;\; 0] & \hat{x} \end{array}\right]\Psi_\perp &\doteq \left[\begin{array}{c|c} B & z \end{array}\right]\\ \Psi_\perp^T\left(\operatorname{diag}(0,0,1)-\Omega(S,T,G)\right)\Psi_\perp &\doteq Q=\left[\begin{array}{c|c} Q_{11} & q_{12} \\ \hline q_{12}^T & 1-q_{22}\end{array}\right],\end{aligned}$$

where $B=[I_n \;\; 0]\Psi_{\perp 1}$, $z=[I_n \;\; 0]\psi_{\perp 2}-\hat{x}$. Notice that $\Psi_\perp^T\operatorname{diag}(0,0,1)\Psi_\perp=\operatorname{diag}(0,0,1)$. Then, condition (2.19) is equivalent to the following condition, obtained by simple reordering of the blocks (dependence on S,T,G is sometimes omitted to avoid clutter)

$$\left[\begin{array}{ccc} P & z & B \\ z^T & 1-q_{22} & q_{12}^T \\ B^T & q_{12} & Q_{11}\end{array}\right]\succeq 0.$$

Now, by Lemma A.7, the above matrix inequality is feasible for some P,z if and only if

$$Q(S,T,G)\succeq 0,\;\left[\begin{array}{cc} P & B \\ B^T & Q_{11}\end{array}\right]\succeq 0 \tag{B.80}$$

is feasible for some P. Therefore problem (2.25) is equivalent to

$$\begin{aligned}&\min_{S,T,G}\min_{P} f(P) \text{ subject to } (B.80),\\ &\quad (S,T,G)\in\mathcal{B}(\boldsymbol{\Delta}),\; S\succeq 0, T\succeq 0,\end{aligned}$$

which, by Corollary A.1, is equivalent to

$$\begin{aligned}&\min_{S,T,G} f(\bar{X}(S,T,G)) \text{ subject to}\\ &\quad (S,T,G)\in\mathcal{B}(\boldsymbol{\Delta}),\; S\succeq 0, T\succeq 0,\\ &\quad Q(S,T,G)\succeq 0,\\ &\quad (I-Q_{11}^\dagger Q_{11})B^T=0,\end{aligned}$$

where $\bar{X}(S,T,G)=BQ_{11}^\dagger(S,T,G)B^T$.

If S_{opt},T_{opt},G_{opt} are the optimal values of the above optimization problem, then (again by Corollary A.1) the optimal ellipsoid is given by

$$\begin{aligned}P_{opt} &= BQ_{11}^\dagger(S_{opt},T_{opt},G_{opt})B^T\\ z_{opt} &= BQ_{11}^\dagger(S_{opt},T_{opt},G_{opt})Q_{12}.\end{aligned}$$

From the latter we then retrieve the ellipsoid center as

$$\hat{x}_{opt}=[I_n \;\; 0]\psi_{\perp 2}-z_{opt}.$$

□

References

[1] A. Ben-Tal and A. Nemirovski. *Lectures on modern convex optimization*. SIAM, 2001.

[2] A. Ben-Tal and A. Nemirovski. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM Journal on Optimization*, 12(3):811–833, 2002.

[3] D.P. Bertsekas and I.B. Rhodes. Recursive state estimation with a set-membership description of the uncertainty. *IEEE Trans. Aut. Control*, 16:117–128, 1971.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, June 1994.

[5] F.L. Chernousko. *State Estimation of Dynamic Systems*. CRC Press, Boca Raton, FL, 1993.

[6] C. Durieu, E. Walter, and B. Polyak. Multi-input multi-output ellipsoidal state bounding. In *Journal of Optimization Theory and Applications*, volume 111, 2001.

[7] L. El Ghaoui and G. Calafiore. Confidence ellipsoids for uncertain linear equations with structure. In *Proc. IEEE Conf. on Decision and Control*, volume 2, pages 1922–1927, Phoenix, AZ, December 1999.

[8] L. El Ghaoui and G. Calafiore. *Worst-case simulation of uncertain systems*, volume 245 of *Lecture Notes in Control and Information Sciences*, pages 134–146. Springer-Verlag, London, 1999.

[9] M.K.H. Fan, A.L. Tits, and J.C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Control*, 36:25–38, 1991.

[10] E. Fogel and Y.F. Huang. On the value of information in system identification - bounded noise case. *Automatica*, 18(2):229–238, 1982.

[11] L. El Ghaoui. Inversion error, condition number, and approximate inverses of uncertain matrices. *Linear Algebra and its Applications*, 342(1-3), February 2002.

[12] L. El Ghaoui and G. Calafiore. Robust filtering for discrete-time systems with bounded noise and parametric uncertainty. *IEEE Trans. Aut. Control*, 46(7):1084–1089, July 2001.

[13] L. El Ghaoui and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM J. Optim.*, 9(1):33–52, 1998.

[14] M.X. Goemans and D.P. Williamson. .878-approximation for MAX CUT and MAX 2SAT. *Proc. 26th ACM Sym. Theor. Computing*, pages 422–431, 1994.

[15] E.R. Hansen. Bounding the solution of interval linear equations. *SIAM J. on Numerical Analysis*, 29:1493–1503, 1992.

[16] D. Henrion, S. Tarbouriech, and D. Arzelier. LMI approximations for the radius of the intersection of ellipsoids. *J. of Optimization Theory and Applications*, 108(1):1–28, 2001.

[17] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, U.K., 1985.

[18] S. Van Huffel and J. Vandewalle. *The Total Least Squares Problem: Computational Aspects and Analysis*. SIAM, Philadelphia, 1991.

[19] U. Köyluoglu, A.S.Cakmak, and S.R.K. Nielsen. Interval algebra to deal with pattern loading and structural uncertainty. *Journal of Engineering Mechanics*, pages 1149–1157, 1995.

[20] A.B. Kurzhanski and I. Valyi. *Ellipsoidal Calculus for Estimation and Control*. Birkhauser, Boston, 1996.

[21] J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001.

[22] D.G. Maskarov and J.P. Norton. State bounding with ellipsoidal set description of the uncertainty. *Int. J. Control*, 65(5):847–866, 1996.

[23] R.L. Muhanna and R.L. Mullen. Uncertainty in mechanics problems - interval-based approach. *ASCE Journal of Engineering Mechanics*, 127(6):557–566, 2001.

[24] A. Nemirovski, C. Roos, and T. Terlaky. On maximization of quadratic form over intersection of ellipsoids with common center. *Math. Program.*, 86(3):463–473, 1999.

[25] Y. Nesterov and A. Nemirovski. *Interior point polynomial methods in convex programming: theory and applications*. SIAM, Philadelphia, 1994.

[26] Yu. Nesterov. Semidefinite relaxation and non-convex quadratic optimization. *Optimization Methods and Software*, 12:1–20, 1997.

[27] A. Neumaier. *Interval Methods for Systems of Equations*. Cambridge University Press, Cambridge, UK, 1990.

[28] J.P. Norton and S.M. Veres. Developments in parameter bounding. In L. Gerencser and P. E. Caines, editors, *Topics in Stochastic Systems: Modelling, Estimation and Adaptive Control*, pages 1137–1158. Springer-Verlag, Berlin, 1991.

[29] A.I. Ovseevich. On equations of ellipsoids approximating attainable sets. *Journal of Optimization Theory and Applications*, 95(3):659–676, 1997.

[30] P.A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. Ph.D. Thesis, California Institute of Technology, 2000.

[31] B.T. Polyak. Convexity of quadratic transformations and its use in control and optimization. *Journal of Optimization Theory and Applications*, 99(3):553–583, 1998.

[32] B.T. Polyak, S.A. Nazin, C. Durieu, and E. Walter. Ellipsoidal estimation under model uncertainty. In *Proceedings of IFAC 2002 World Congress*, Barcelona, Spain, 2002.

[33] J. Rohn. Systems of linear interval equations. *Linear Algebra and its Applications*, 126:39–78, 1989.

[34] J. Rohn. Overestimations in bounding solutions of perturbed linear equations. *Linear Algebra and its Applications*, 262:55–66, 1997.

[35] F.C. Schweppe. Recursive state estimation: unknown but bounded errors and system inputs. *IEEE Trans. Aut. Control*, 13(1):22–28, 1968.

[36] F.C. Schweppe. *Uncertain Dynamic Systems*. Prentice-Hall, New York, 1973.

[37] J.S. Shamma and K.-Y. Tu. Set-valued observers and optimal disturbance rejection. *IEEE Trans. Aut. Control*, 44(2):253–264, 1999.

[38] R.E. Skelton, T. Iwasaki, and K. Grigoriadis. *A unified algebraic approach to linear control design*. Taylor & Francis, London, 1998.

[39] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, March 1996.

[40] L. Vandenberghe, S. Boyd, and S.-P. Wu. Determinant maximization with linear matrix inequality constraints. *SIAM J. on Matrix Analysis and Applications*, 19(2):499–533, April 1998.

[41] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, 1996.