

# Parameter Estimation with Expected and Residual-at-Risk Criteria

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## Abstract

In this paper we study a class of uncertain linear estimation problems in which the data are affected by random uncertainty. In this setting, we consider two estimation criteria, one based on minimization of the expected  $\ell_1$  or  $\ell_2$  norm residual and one based on minimization of the level within which the  $\ell_1$  or  $\ell_2$  norm residual is guaranteed to lie with an a-priori fixed probability (residual at risk). The random uncertainty affecting the data is characterized by means of its first two statistical moments, and the above criteria are intended in a worst-case probabilistic sense, that is worst-case expectations and probabilities over all possible distribution having the specified moments are considered. The ensuing estimation problems can be solved efficiently via convex programming, yielding exact solutions in the  $\ell_2$  norm case and upper-bounds on the optimal solutions in the  $\ell_1$  case.

*Keywords:* Uncertain least-squares, random uncertainty, robust convex optimization, value at risk,  $\ell_1$  norm approximation.

## 1 Introduction

To introduce the problem treated in this paper, let us first consider a standard parameter estimation problem where an unknown parameter  $\theta \in \mathbb{R}^n$  is to be determined so to minimize a norm residual of the form  $\|A\theta - b\|_p$ , where  $A \in \mathbb{R}^{m,n}$  is a given regression matrix,  $b \in \mathbb{R}^m$  is a measurement vector, and  $\|\cdot\|_p$  denotes the  $\ell_p$  norm. In this setting, the most relevant and widely studied case arise of course for  $p = 2$ , where the problem reduces to classical least-squares. The case of  $p = 1$  also has important applications due to its resilience to outliers and to the property of producing “sparse” solutions, see for instance [5, 7]. For  $p = 1$ , the solution to the norm minimization problem can be efficiently computed via linear programming, [3, §6.2].

In this paper we are concerned with an extension of this basic setup that arises in realistic cases where the problem data  $A, b$  are imprecisely known. Specifically, we consider the situation where the entries of  $A, b$  depend affinely on a vector  $\delta$  of random uncertain parameters, that is  $A \doteq A(\delta)$  and  $b \doteq b(\delta)$ . Due its practical significance, the parameter estimation problem in the presence of uncertainty in the data has attracted much attention

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in the literature. When the uncertainty is modeled as unknown-but-bounded, a min-max approach is followed in [8], where the maximum over the uncertainty of the  $\ell_2$  norm of the residual is minimized. Relations between the min-max approach and regularization techniques are also discussed in [8] and in [12]. Generalizations of this approach to  $\ell_1$  and  $\ell_\infty$  norms are proposed in [10].

In the case when the uncertainty is assumed to be random with given distribution, a classical stochastic optimization approach is often followed, whereby a  $\theta$  is sought that minimizes the expectation of the  $\ell_p$  norm of the residual with respect to the uncertainty. This formulation leads in general to numerically “hard” problem instances, that can be solved approximately by means of stochastic approximation methods, see, e.g., [4]. In the special case where the squared Euclidean norm is considered, instead, the expected value minimization problem actually reduces to a standard least-squares problem, which has a closed-form solution, see [4, 10].

In this paper we consider the uncertainty to be random and we develop our results in a “statistical ambiguity” setting in which the probability distribution of the uncertainty is only known to belong to a given family of distributions. Specifically, we consider the family of all distributions on the uncertainty having a given mean and covariance, and seek results that are guaranteed irrespective of the actual distribution within this class. We address both the  $\ell_2$  and  $\ell_1$  cases, under two different estimation criteria: the first criterion aims at minimizing the worst-case expected residual, whereas the second one is directly tailored to control residual tail probabilities. That is, for given risk  $\epsilon \in (0, 1)$ , we minimize the residual level such that the probability of residual falling above this level is no larger than  $\epsilon$ .

The rest of the paper is organized as follows: Section 2 sets up the problem and gives some preliminary results. Section 3 is devoted to the estimation scheme with worst-case expected residual criterion, while Section 4 contains the results for the residual-at-risk criterion. Numerical examples are presented in Section 5 and conclusions are drawn in Section 6. An Appendix contains some of the technical derivations.

**Notation** The identity matrix in  $\mathbb{R}^{n,n}$  and the zero matrix in  $\mathbb{R}^{n,n}$  are denoted as  $I_n$  and  $0_n$ , respectively (subscripts may be omitted when they can easily be inferred from context).  $\|x\|_p$  denotes the standard  $\ell_p$  norm of vector  $x$ ;  $\|X\|_F$  denotes the Frobenius norm of matrix  $X$ , that is  $\|X\|_F = \sqrt{\text{Tr } X^\top X}$ , where  $\text{Tr}$  is the trace operator. The notation  $\delta \sim (\hat{\delta}, D)$  means that  $\delta$  is a random vector with expected value  $\mathbb{E}\{\delta\} = \hat{\delta}$  and covariance matrix  $\text{var}\{\delta\} \doteq \mathbb{E}\{(\delta - \hat{\delta})(\delta - \hat{\delta})^\top\} = D$ . The notation  $X \succ 0$  (resp.  $X \succeq 0$ ) indicates that matrix  $X$  is symmetric and positive definite (resp. semi-definite).

## 2 Problem setup and preliminaries

Let  $A(\delta) \in \mathbb{R}^{m,n}$ ,  $b(\delta) \in \mathbb{R}^m$  be such that

$$[A(\delta) \ b(\delta)] \doteq [A_0 \ b_0] + \sum_{i=1}^q \delta_i [A_i \ b_i], \quad (1)$$

where  $\delta = [\delta_1 \ \cdots \ \delta_q]^\top$  is a vector random uncertainties,  $[A_0 \ b_0]$  represents the “nominal” data, and  $[A_i \ b_i]$  are the matrices of coefficients for the uncertain part of the data. Let

$\theta \in \mathbb{R}^n$  be a parameter to be estimated, and consider the following norm residual (which is a function of both  $\theta$  and  $\delta$ ):

$$\begin{aligned} f_p(\theta, \delta) &\doteq \|A(\delta)\theta - b(\delta)\|_p \\ &= \|[(A_1\theta - b_1) \cdots (A_q\theta - b_q)]\delta + (A_0\theta - b_0)\|_p \\ &\doteq \|L(\theta)z\|_p, \end{aligned} \quad (2)$$

where we defined  $z \doteq [\delta^\top \ 1]^\top$ , and  $L(\theta) \in \mathbb{R}^{m, q+1}$  is partitioned as

$$L(\theta) \doteq [L^{(\delta)}(\theta) \ L^{(1)}(\theta)], \quad (3)$$

with

$$L^{(\delta)}(\theta) \doteq [(A_1\theta - b_1) \cdots (A_q\theta - b_q)] \in \mathbb{R}^{m, q}, \quad L^{(1)}(\theta) \doteq A_0\theta - b_0 \in \mathbb{R}^m. \quad (4)$$

In the following we assume that  $\mathbb{E}\{\delta\} = 0$  and  $\text{var}\{\delta\} = I_q$ . This can be done without loss of generality, since data can always be pre-processed so to comply with this assumption, as detailed in the following remark.

**Remark 1 (Preprocessing the data)** Suppose that the uncertainty  $\delta$  is such that  $\mathbb{E}\{\delta\} = \hat{\delta}$  and  $\text{var}\{\delta\} = D \succeq 0$ , and let  $D = QQ^\top$  be a full-rank factorization of  $D$ . Then, we may write  $\delta = Q\nu + \hat{\delta}$ , with  $\mathbb{E}\{\nu\} = 0$ ,  $\text{var}\{\nu\} = I_q$ , and redefine the problem in terms of uncertainty  $\nu \sim (0, I)$ , with  $L^{(\delta)}(\theta) = [(A_1\theta - b_1) \cdots (A_q\theta - b_q)]Q$ ,  $L^{(1)}(\theta) = [(A_1\theta - b_1) \cdots (A_q\theta - b_q)]\hat{\delta} + (A_0\theta - b_0)$ .  $\triangleleft$

We next state the two estimation criteria and the ensuing problems that are tackled in this paper.

**Problem 1 (Worst-case expected residual minimization)** Determine  $\theta \in \mathbb{R}^n$  that minimizes  $\sup_{\delta \sim (0, I)} \mathbb{E}\{f_p(\theta, \delta)\}$ , that is solve

$$\min_{\theta \in \mathbb{R}^n} \sup_{\delta \sim (0, I)} \mathbb{E}\{\|L(\theta)z\|_p\}, \quad z^\top \doteq [\delta^\top \ 1], \quad (5)$$

where  $p \in \{1, 2\}$ ,  $L(\theta)$  is given in (3), (4), and the supremum is taken with respect to all possible probability distributions having the specified moments (zero mean and unit covariance).

In some applications, such as in financial Value-at-Risk (V@R) [6, 11], one is interested in guaranteeing that the residual remains “small” in “most” of the cases, that is one seeks  $\theta$  such that the corresponding residual is small with high probability. An expected residual criterion such as the one considered in Problem 1 is not suitable for this purpose, since it concentrates on the average case, neglecting the tails of the residual distribution. The second criterion that we consider is hence focused on controlling the risk of having residuals above some level  $\gamma \geq 0$ , where risk is expressed as the probability  $\text{Prob}\{\delta : f(\theta, \delta) \geq \gamma\}$ . Formally, we state the following second problem.

**Problem 2 (Guaranteed residual-at-risk minimization)** Fix a risk level  $\epsilon \in (0, 1)$ . Determine  $\theta \in \mathbb{R}^n$  such that a residual level  $\gamma$  is minimized while guaranteeing that  $\text{Prob}\{\delta : f_p(\theta, \delta) \geq \gamma\} \leq \epsilon$ . That is, solve

$$\begin{aligned} &\min_{\theta \in \mathbb{R}^n, \gamma \geq 0} && \gamma \\ \text{subject to : } & \sup_{\delta \sim (0, I)} \text{Prob}\{\delta : \|L(\theta)z\|_p \geq \gamma\} \leq \epsilon, && z^\top \doteq [\delta^\top \ 1], \end{aligned}$$

where  $p \in \{1, 2\}$ ,  $L(\theta)$  is given in (3), (4), and the supremum is taken with respect to all possible probability distributions having the specified moments (zero mean and unit covariance).

A key preliminary result opening the way for the solution of Problem 1 and Problem 2 is stated in the next lemma. This lemma is a powerful consequence of convex duality, and provides a general result for computing the supremum of expectations and probabilities over all distributions possessing a given mean and covariance matrix.

**Lemma 1** *Let  $S \subseteq \mathbb{R}^n$  be a measurable set (not necessarily convex), and  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  a measurable function. Let  $z^\top = [x^\top \ 1]$ , and define*

$$\begin{aligned} E_{\text{wc}} &\doteq \sup_{x \sim (\hat{x}, \Gamma)} \mathbb{E} \{ \phi(x) \} \\ P_{\text{wc}} &\doteq \sup_{x \sim (\hat{x}, \Gamma)} \text{Prob} \{ x \in S \} \\ Q &\doteq \begin{bmatrix} \Gamma + \hat{x}\hat{x}^\top & \hat{x} \\ \hat{x}^\top & 1 \end{bmatrix}. \end{aligned}$$

Then,

$$E_{\text{wc}} = \inf_{M=M^\top} \text{Tr} QM \quad \text{subject to: } z^\top Mz \geq \phi(x), \quad \forall x \in \mathbb{R}^n \quad (6)$$

and

$$P_{\text{wc}} = \inf_{M \succeq 0} \text{Tr} QM \quad \text{subject to: } z^\top Mz \geq 1, \quad \forall x \in S. \quad (7)$$

◁

A proof of Lemma 1 is given in the Appendix.

**Remark 2** Lemma 1 provides a result for computing worst-case expectations and probabilities. However in many cases of interest we shall need to impose constraints on these quantities in order to eventually optimize them with respect to some other design variables. In this respect, the following equivalence holds:

$$\sup_{x \sim (\hat{x}, \Gamma)} \mathbb{E} \{ \phi(x) \} \leq \gamma \quad (8)$$

⇕

$$\exists M = M^\top : \text{Tr} QM \leq \gamma, \quad z^\top Mz \geq \phi(x), \quad \forall x \in \mathbb{R}^n. \quad (9)$$

To verify this fact, consider first the ⇓ direction: If (8) holds, we let  $M$  be the solution that achieves the optimum in (6), and we have that (9) holds. On the other hand, if (9) holds for some  $M = \bar{M}$ , then  $\sup_{x \sim (\hat{x}, \Gamma)} \mathbb{E} \{ \phi(x) \} = \inf \text{Tr} QM \leq \text{Tr} Q\bar{M} \leq \gamma$ , which concludes proves the statement. By an analogous reasoning, we can verify that

$$\sup_{x \sim (\hat{x}, \Gamma)} \text{Prob} \{ x \in S \} \leq \epsilon$$

⇕

$$\exists M = M^\top : M \succeq 0, \quad \text{Tr} QM \leq \epsilon, \quad z^\top Mz \geq 1, \quad \forall x \in S.$$

◁

### 3 Worst-case expected residual minimization

In this section we focus on Problem 1 and provide an efficiently computable exact solution for the case  $p = 2$ , and efficiently computable upper and lower bounds on the solution for the case  $p = 1$ . Define

$$\psi_p(\theta) \doteq \sup_{\delta \sim (0, I)} \mathbb{E} \{ \|L(\theta)z\|_p \}, \quad \text{with } z^\top \doteq [\delta^\top \ 1], \quad (10)$$

$$r \doteq [0 \ \cdots \ 0 \ 1/2]^\top \in \mathbb{R}^{q+1}, \quad (11)$$

where  $L(\theta) \in \mathbb{R}^{m, q+1}$  is an affine function of parameter  $\theta$ , given in (3), (4). We have the following preliminary lemma.

**Lemma 2** *For given  $\theta \in \mathbb{R}^n$ , the worst-case residual expectation  $\psi_p(\theta)$  is given by*

$$\begin{aligned} \psi_p(\theta) = \inf_{M=M^\top} \text{Tr } M \\ \text{subject to: } M - ru^\top L(\theta) - L(\theta)^\top ur^\top \succeq 0, \quad \forall u \in \mathbb{R}^m : \|u\|_{p^*} \leq 1, \end{aligned}$$

where  $\|u\|_{p^*}$  is the dual  $\ell_p$  norm. ◁

**Proof.** From Lemma 1 we have that

$$\begin{aligned} \psi_p(\theta) = \inf_{M=M^\top} \text{Tr } M \\ \text{subject to: } z^\top Mz \geq \|L(\theta)z\|_p, \quad \forall \delta \in \mathbb{R}^q. \end{aligned}$$

Since

$$\|L(\theta)z\|_p = \sup_{\|u\|_{p^*} \leq 1} u^\top L(\theta)z,$$

it follows that  $z^\top Mz \geq \|L(\theta)z\|_p$  holds for all  $\delta$  if and only if

$$z^\top Mz \geq u^\top L(\theta)z, \quad \forall \delta \in \mathbb{R}^q \text{ and } \forall u \in \mathbb{R}^m : \|u\|_{p^*} \leq 1.$$

Now, since  $z^\top r = 1/2$ , we write  $u^\top L(\theta)z = z^\top (ru^\top L(\theta) + L(\theta)^\top ur^\top)z$ , whereby the above condition is satisfied if and only if

$$M - ru^\top L(\theta) + L(\theta)^\top ur^\top \succeq 0, \quad \forall u : \|u\|_{p^*} \leq 1,$$

which concludes the proof. ◻

We are now in position to state the following key theorem.

**Theorem 1** *Let  $\theta \in \mathbb{R}^n$  be given, and let  $\psi_p(\theta)$  be defined as in (10). Then, the following holds for the worst-case expected residuals in the  $\ell_1$ - and  $\ell_2$ -norm cases.*

1. *Case  $p = 1$ : Define*

$$\bar{\psi}_1(\theta) \doteq \sum_{i=1}^m \|L_i(\theta)^\top\|_2, \quad (12)$$

where  $L_i(\theta)^\top$  denotes the  $i$ -th row of  $L(\theta)$ . Then,

$$\frac{2}{\pi} \bar{\psi}_1(\theta) \leq \psi_1(\theta) \leq \bar{\psi}_1(\theta). \quad (13)$$

2. Case  $p = 2$ :

$$\psi_2(\theta) = \sqrt{\text{Tr } L(\theta)^\top L(\theta)} = \|L(\theta)\|_F. \quad (14)$$

◁

**Proof.** (Case  $p = 1$ ) The dual  $\ell_1$  norm is the  $\ell_\infty$  norm, hence applying Lemma 2 we have

$$\begin{aligned} \psi_1(\theta) = & \inf_{M=M^\top} \text{Tr } M \\ \text{subject to: } & M - L(\theta)^\top ur^\top - ru^\top L(\theta) \succeq 0, \quad \forall u : \|u\|_\infty \leq 1. \end{aligned} \quad (15)$$

For ease of notation, we drop the dependence on  $\theta$  in the following derivation. Note that

$$L^\top ur^\top + ru^\top L = \sum_{i=1}^m u_i C_i,$$

where

$$C_i \doteq rL_i^\top + L_i r^\top = \begin{bmatrix} 0_q & \frac{1}{2}L_i^{(\delta)} \\ \frac{1}{2}L_i^{(\delta)\top} & L_i^{(1)} \end{bmatrix} \in \mathbb{R}^{q+1, q+1},$$

where  $L_i^\top$  is partitioned according to (4) as  $L_i^\top = [L_i^{(\delta)\top} \ L_i^{(1)}]$ , with  $L_i^{(\delta)\top} \in \mathbb{R}^{1, q}$ , and  $L_i^{(1)} \in \mathbb{R}$ . The characteristic polynomial of  $C_i$  is  $p_i(s) = s^{q-1}(s^2 - L_i^{(1)}s - \|L_i^{(\delta)}\|_2^2/4)$ , hence  $C_i$  has  $q - 1$  null eigenvalues, and two non-zero eigenvalues at  $\eta_{i,1} = (L_i^{(1)} + \|L_i^{(\delta)}\|_2)/2 > 0$ ,  $\eta_{i,2} = (L_i^{(1)} - \|L_i^{(\delta)}\|_2)/2 < 0$ . Since  $C_i$  is rank two, the constraint in problem (15) takes the form (23) considered in Theorem 4 in the Appendix. Consider thus the following relaxation of problem (15):

$$\begin{aligned} \varphi \doteq & \inf_{M=M^\top, X_i=X_i^\top} \text{Tr } M \\ \text{subject to: } & -X_i + C_i \preceq 0, \quad -X_i - C_i \preceq 0, \quad i = 1, \dots, m, \\ & \sum_{i=1}^m X_i - M \preceq 0, \end{aligned} \quad (16)$$

where we clearly have  $\psi_1 \leq \varphi$ . The dual of problem (16) can be written as

$$\begin{aligned} \varphi^D = & \sup_{\Lambda_i, \Gamma_i} \sum_{i=1}^m \text{Tr} ((\Lambda_i - \Gamma_i)C_i) \\ \text{subject to: } & \Lambda_i + \Gamma_i = I_{q+1}, \\ & \Gamma_i \succeq 0, \quad \Lambda_i \succeq 0, \quad i = 1, \dots, m. \end{aligned} \quad (17)$$

Since the problem in (16) is convex and Slater conditions are satisfied,  $\varphi = \varphi^D$ . Next we show that  $\varphi^D$  equals  $\bar{\psi}_1$  given in (12). To this end, observe that (17) is decoupled in the  $\Gamma_i, \Lambda_i$  variables and, for each  $i$ , the subproblem amounts to determining  $\sup_{0 \preceq \Gamma_i \preceq I} \text{Tr} (I - 2\Gamma_i)C_i$ . By diagonalizing  $C_i$  as  $C_i = V_i \Theta_i V_i^\top$ , with  $\Theta_i = \text{diag}(0, \dots, 0, \eta_{i,1}, \eta_{i,2})$ , each subproblem is reformulated as  $\sup_{0 \preceq \tilde{\Gamma}_i \preceq I} \text{Tr } C_i - 2\text{Tr } \Theta_i \tilde{\Gamma}_i$ , where it immediately follows that the optimal solution is  $\tilde{\Gamma}_i = \text{diag}(0, \dots, 0, 0, 1)$ , hence the supremum is  $(\eta_{i,1} + \eta_{i,2}) - 2\eta_{i,2} = \eta_{i,1} - \eta_{i,2} = |\eta_{i,1}| + |\eta_{i,2}| = \|\text{eig}(C_i)\|_1$ , where  $\text{eig}(\cdot)$  denotes the vector of the eigenvalues of its argument. Now, we have  $\|\text{eig}(C_i)\|_1 = \|L_i^\top\|_2$ , then  $\varphi^D = \sum_{i=1}^m \|L_i^\top\|_2$ , and by the first conclusion in Theorem 4 in the Appendix, we have  $\bar{\psi}_1 = \varphi = \varphi^D$  and  $\psi_1 \leq \bar{\psi}_1$ .

For the lower bound on  $\psi_1$  in (13), assume that the problem in (17) is not feasible. Then, for  $M \succeq 0$ , we have that

$$\begin{aligned} & \{M : \text{Tr } M = \varphi^D\} \cap \\ & \{M : X_i \succeq \pm C_i, \quad \sum_{i=1}^m X_i \preceq M\} = \emptyset. \end{aligned}$$

This last emptiness statement, coupled with the fact that, for  $i = 1, \dots, n$ ,  $C_i$  is of rank two, implies, by the second conclusion in Theorem 4, that

$$\left\{ M : \text{Tr } M = \varphi^D \right\} \cap \left\{ M : M \succeq \sum_{i=1}^n u_i C_i, \forall u : |u_i| \leq \pi/2 \right\} = \emptyset$$

and

$$\left\{ \tilde{M} : \text{Tr } \tilde{M} = \frac{\varphi^D}{\pi/2} \right\} \cap \left\{ \tilde{M} : \tilde{M} \succeq \sum_{i=1}^n \tilde{u}_i C_i, \forall \tilde{u} : |\tilde{u}_i| \leq 1 \right\} = \emptyset$$

Consequently, we have  $\psi_1 \geq \frac{\varphi^D}{\pi/2} = \frac{\bar{\psi}_1}{\pi/2}$ , which concludes the proof of the  $p = 1$  case.

(Case  $p = 2$ ) The dual  $\ell_2$  norm is the  $\ell_2$  norm itself, hence applying Lemma 2 we have

$$\psi_2 = \inf_{M=M^\top} \text{Tr } M \quad \text{subject to: } M - ru^\top L - L^\top ur^\top \succeq 0, \quad \forall u : \|u\|_2 \leq 1.$$

Applying the LMI robustness lemma (Lemma 3.1 of [9]), we have that the previous semi-infinite problem is equivalent to the following SDP

$$\psi_2(\theta) = \inf_{M=M^\top, \tau > 0} \text{Tr } M \quad \text{subject to: } \begin{bmatrix} M - \tau r r^\top & L^\top \\ L & \tau I_m \end{bmatrix} \succeq 0.$$

By the Schur complement rule, the latter constraint is equivalent to  $\tau > 0$  and  $M \succeq \frac{1}{\tau}(L^\top L) + \tau r r^\top$ . Thus, the infimum of  $\text{Tr } M$  is achieved for  $M = \frac{1}{\tau}(L^\top L) + \tau r r^\top$  and, since  $r r^\top = \text{diag}(0_q, 1/4)$ , the infimum of  $\text{Tr } M$  over  $\tau > 0$  is achieved for  $\tau = 2\sqrt{\text{Tr } L^\top L}$ . From this, it follows that  $\psi_2 = \sqrt{\text{Tr } L^\top L}$ , thus concluding the proof.  $\square$

Starting from the results in Theorem 1, it is easy to observe that we can further minimize the residuals over the parameter  $\theta$ , in order to find a solution to Problem 1. Convexity of the ensuing minimization problem is a consequence of the fact that  $L(\theta)$  is an affine function of  $\theta$ . This is formalized in the following corollary, whose simple proof is omitted.

**Corollary 1 (Worst-case expected residual minimization)** *Let*

$$\psi_p^* \doteq \min_{\theta \in \mathbb{R}^n} \sup_{\delta \sim (0, I)} \text{E} \{ \|L(\theta)z\|_p \}, \quad z^\top \doteq [\delta^\top \ 1].$$

For  $p = 1$ , it holds that

$$\frac{2}{\pi} \bar{\psi}_1^* \leq \psi_1^* \leq \bar{\psi}_1^*,$$

where  $\bar{\psi}_1^*$  is computed by solving the following second-order-cone (SOCP) program:

$$\bar{\psi}_1^* = \min_{\theta \in \mathbb{R}^n} \sum_{i=1}^m \|L_i(\theta)^\top\|_2.$$

For  $p = 2$ , it holds that

$$\psi_2^* = \min_{\theta \in \mathbb{R}^n} \|L(\theta)\|_F,$$

where a minimizer for this problem can be computed via convex quadratic programming, by minimizing  $\text{Tr } L^\top(\theta)L(\theta)$ .  $\triangleleft$

**Remark 3** Notice that in the specific case of  $\delta \sim (0, I)$  we have that  $\psi_2^2 = \text{Tr } L^\top(\theta)L(\theta) = \sum_{i=0}^q \|A_i\theta - b_i\|_2^2$ , hence the minimizer can in this case be determined by standard Least-Squares solution method. Interestingly, this solution coincides with the solution of the expected *squared*  $\ell_2$ -norm minimization problem discussed for instance in [4, 10]. This might not be obvious, since in general  $\text{E}\{\|\cdot\|^2\} \neq (\text{E}\{\|\cdot\|\})^2$ .  $\triangleleft$

## 4 Guaranteed residual-at-risk minimization

### 4.1 The $\ell_2$ -norm case

Assume first  $\theta \in \mathbb{R}^n$  is fixed, and consider the problem of computing

$$P_{\text{wc}_2}(\theta) = \sup_{\delta \sim (0, I)} \text{Prob} \{ \delta : \|L(\theta)z\|_2 \geq \gamma \} = \sup_{\delta \sim (0, I)} \text{Prob} \{ \delta : \|L(\theta)z\|_2^2 \geq \gamma^2 \},$$

where  $z^\top \doteq [\delta^\top \ 1]$ . By Lemma 1, this probability corresponds to the optimal value of the optimization problem

$$\begin{aligned} P_{\text{wc}_2}(\theta) &= \inf_{M \succeq 0} \text{Tr } M \\ &\text{subject to: } z^\top M z \geq 1, \quad \forall \delta : \|L(\theta)z\|_2^2 \geq \gamma^2, \end{aligned}$$

where the constraint can be written equivalently as

$$z^\top (M - \text{diag}(0_q, 1)) z \geq 0, \quad \forall \delta : z^\top (L(\theta)^\top L(\theta) - \text{diag}(0_q, \gamma^2)) z \geq 0.$$

Applying the lossless  $S$ -procedure, the condition above is in turn equivalent to the existence of  $\tau \geq 0$  such that  $(M - \text{diag}(0_q, 1)) \succeq \tau (L(\theta)^\top L(\theta) - \text{diag}(0_q, \gamma^2))$ , therefore we obtain

$$\begin{aligned} P_{\text{wc}_2}(\theta) &= \inf_{M \succeq 0, \tau > 0} \text{Tr } M \\ &\text{subject to: } M \succeq \tau L(\theta)^\top L(\theta) + \text{diag}(0_q, 1 - \tau\gamma^2), \end{aligned}$$

where the latter expression can be further elaborated using the Schur complement formula into

$$\begin{bmatrix} M - \text{diag}(0_q, 1 - \tau\gamma^2) & \tau L(\theta)^\top \\ \tau L(\theta) & \tau I_m \end{bmatrix} \succeq 0. \quad (18)$$

We now notice, by the reasoning in Remark 2, that the condition  $P_{\text{wc}_2}(\theta) \leq \epsilon$  with  $\epsilon \in (0, 1)$  is equivalent to the conditions

$$\exists \tau \geq 0, M \succeq 0 \text{ such that: } \text{Tr } M \leq \epsilon, \text{ and (18) holds.}$$

A parameter  $\theta$  that minimizes the residual-at-risk level  $\gamma$  while satisfying the condition  $P_{\text{wc}_2}(\theta) \leq \epsilon$  can thus be computed by solving a sequence of convex semidefinite optimization problems parameterized in  $\tau$ , as formalized in the next theorem.

**Theorem 2 ( $\ell_2$  residual-at-risk estimation)** *A solution of Problem 2 in the  $\ell_2$  case can be found by solving the following optimization problem:*

$$\begin{aligned} \inf_{\tau > 0} \inf_{M \succeq 0, \theta \in \mathbb{R}^n, \gamma^2 \in \mathbb{R}^n} & \gamma^2, \\ \text{subject to:} & \text{Tr } M \leq \epsilon \\ & \begin{bmatrix} M - \text{diag}(0_q, 1 - \tau\gamma^2) & \tau L(\theta)^\top \\ \tau L(\theta) & \tau I_m \end{bmatrix} \succeq 0. \end{aligned} \quad (19)$$

$\triangleleft$



**Remark 4** The optimization problem in Theorem 2 is not jointly convex in all variables, due to the presence of product terms  $\tau\gamma^2$ ,  $\tau L(\theta)$ . However, the problem is convex in  $M, \theta, \gamma^2$  for each fixed  $\tau > 0$ , whence an optimal solution can be computed via a line search over  $\tau > 0$ , where each step requires solving an SDP in  $M, \theta$ , and  $\gamma^2$ .  $\triangleleft$

## 4.2 The $\ell_1$ -norm case

We next consider the problem of determining  $\theta \in \mathbb{R}^n$  such that the residual-at-risk level  $\gamma$  is minimized while guaranteeing that  $P_{\text{wc}_1}(\theta) \leq \epsilon$ , where  $P_{\text{wc}_1}(\theta)$  is the worst-case  $\ell_1$ -norm residual tail probability

$$P_{\text{wc}_1}(\theta) = \sup_{\delta \sim (0, I)} \text{Prob} \{ \delta : \|L(\theta)z\|_1 \geq \gamma \},$$

and  $\epsilon \in (0, 1)$  is the a-priori fixed risk level. To this end, define

$$\mathcal{D} \doteq \{ D \in \mathbb{R}^{m, m} : D \text{ diagonal}, D \succ 0 \}$$

and consider the following proposition (whose statement may be easily proven by taking the gradient with respect to  $D$  and setting it to zero).

**Proposition 1** For any  $v \in \mathbb{R}^m$ , it holds that

$$\|v\|_1 = \frac{1}{2} \inf_{D \in \mathcal{D}} \sum_{i=1}^m \left( \frac{v_i^2}{d_i} + d_i \right) = \frac{1}{2} \inf_{D \in \mathcal{D}} (v^\top D^{-1} v + \text{Tr } D), \quad (20)$$

where  $d_i$  is the  $i$ -th diagonal entry of  $D$ .  $\triangleleft$

The following key theorem holds.

**Theorem 3 ( $\ell_1$  residual-at-risk estimation)** Consider the following optimization problem:

$$\begin{aligned} \inf_{\tau > 0} \inf_{M \succeq 0, D \in \mathcal{D}, \theta \in \mathbb{R}^n, \gamma \geq 0} \quad & \gamma \\ \text{subject to:} \quad & \text{Tr } M \leq \epsilon \\ & \begin{bmatrix} M - (1 - 2\tau\gamma + \text{Tr } D)J & \tau L(\theta)^\top \\ \tau L(\theta) & D \end{bmatrix} \succeq 0, \end{aligned} \quad (21)$$

with  $J \doteq \text{diag}(0_q, 1)$ . The optimal value of this program provides an upper bound for Problem 2 in the  $\ell_1$  case, that is an upper bound on the minimum level  $\gamma$  for which there exist  $\theta$  such that  $P_{\text{wc}_1}(\theta) \leq \epsilon$ .  $\triangleleft$

**Remark 5** Similar to the problem (19) in Theorem 2, the above optimization problem is non convex, due to product terms between  $\tau$  and  $\gamma, \theta$ . The problem can however be easily solved numerically via a line search over the scalar  $\tau > 0$ , where each step of the line search involves the solution of an SDP problem in the variables  $M, D, \theta$ , and  $\gamma$ .  $\triangleleft$

**Proof.** Define

$$\begin{aligned} S & \doteq \{ \delta : \|L(\theta)z\|_1 \geq \gamma \} \\ S(D) & \doteq \{ \delta : z^\top L(\theta)^\top D^{-1} L(\theta)z + \text{Tr } D \geq 2\gamma, D \in \mathcal{D} \}. \end{aligned}$$

For ease of notation we drop the dependence on  $\theta$  in the following derivation. Using (20) we have that, for any  $D \in \mathcal{D}$ ,

$$2\|Lz\|_1 \leq z^\top L^\top D^{-1}Lz + \text{Tr } D,$$

hence  $\delta \in S$  implies  $\delta \in S(D)$ , thus  $S \subseteq S(D)$ , for any  $D \in \mathcal{D}$ . This in turn implies that

$$\text{Prob}\{\delta \in S\} \leq \text{Prob}\{\delta \in S(D)\} \leq \sup_{\delta \sim (0,I)} \text{Prob}\{\delta \in S(D)\}$$

for any probability measure and any  $D \in \mathcal{D}$ , and therefore

$$P_{\text{wc1}} = \sup_{\delta \sim (0,I)} \text{Prob}\{\delta \in S\} \leq \inf_{D \in \mathcal{D}} \sup_{\delta \sim (0,I)} \text{Prob}\{\delta \in S(D)\} \doteq \bar{P}_{\text{wc1}}.$$

Note that, for fixed  $D \in \mathcal{D}$ , we can compute  $P_{\text{wc1}}(D) \doteq \sup_{\delta \sim (0,I)} \text{Prob}\{\delta \in S(D)\}$  from its equivalent dual:

$$\begin{aligned} P_{\text{wc1}}(D) &= \inf_{M \succeq 0} \text{Tr } M : z^\top Mz \geq 1, \forall \delta \in S(D) \\ &= \inf_{M \succeq 0} \text{Tr } M : z^\top Mz \geq 1, \forall \delta : z^\top L^\top D^{-1}Lz + \text{Tr } D \geq 2\gamma \\ &\quad [\text{applying the lossless S-procedure}] \\ &= \inf_{M \succeq 0, \tau > 0} \text{Tr } M : M \succeq \tau L^\top D^{-1}L + (1 - 2\tau\gamma + \tau \text{Tr } D)J, \end{aligned}$$

where  $J = \text{diag}(0_q, 1)$ . Hence,  $\bar{P}_{\text{wc1}}$  is obtained by minimizing  $P_{\text{wc1}}(D)$  over  $D \in \mathcal{D}$ , which results in

$$\begin{aligned} \bar{P}_{\text{wc1}} &= \inf_{M \succeq 0, \tau > 0, D \in \mathcal{D}} \text{Tr } M : M \succeq \tau L^\top D^{-1}L + (1 - 2\tau\gamma + \tau \text{Tr } D)J \\ &\quad [\text{by change of variable } \tau D \rightarrow D \text{ (whence } \tau \text{Tr } D \rightarrow \text{Tr } D, \tau D^{-1} \rightarrow \tau^2 D^{-1})] \\ &= \inf_{M \succeq 0, \tau > 0, D \in \mathcal{D}} \text{Tr } M : M \succeq \tau^2 L^\top D^{-1}L + (1 - 2\tau\gamma + \text{Tr } D)J \\ &= \inf_{M \succeq 0, \tau > 0, D \in \mathcal{D}} \text{Tr } M : \begin{bmatrix} M - (1 - 2\tau\gamma + \text{Tr } D)J & \tau L^\top \\ \tau L & D \end{bmatrix} \succeq 0. \end{aligned}$$

Now, from the reasoning in Remark 2, we have that (re-introducing the dependence on  $\theta$  in the notation)  $\bar{P}_{\text{wc1}}(\theta) \leq \epsilon$  if and only if there exist  $M \succeq 0$ ,  $\tau > 0$  and  $D \in \mathcal{D}$  such that  $\text{Tr } M \leq \epsilon$  and

$$\begin{bmatrix} M - (1 - 2\tau\gamma + \text{Tr } D)J & \tau L(\theta)^\top \\ \tau L(\theta) & D \end{bmatrix} \succeq 0. \quad (22)$$

Notice that, since  $L(\theta)$  is affine in  $\theta$ , condition (22) is an LMI in  $M, D, \theta, \gamma$ , for fixed  $\tau$ . We can thus minimize the residual level  $\gamma$  subject to the condition  $\bar{P}_{\text{wc1}}(\theta) \leq \epsilon$  (which implies  $P_{\text{wc1}}(\theta) \leq \epsilon$ ), and this results in the statement of the theorem.  $\square$

## 5 Numerical examples

### 5.1 Worst-case expected residual minimization

As a first example, we use data from a numerical test appeared in [4]. Let

$$A(\delta) = A_0 + \sum_{i=1}^3 \delta_i A_i, \quad b^T = [0 \quad 2 \quad 1 \quad 3],$$

$$\text{with } A_0 = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 1 \\ -2 & 5 & 3 \\ 1 & 4 & 5.2 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and let  $\delta_i$  be independent random perturbations with zero mean and standard deviations  $\sigma_1 = 0.067$ ,  $\sigma_2 = 0.1$ ,  $\sigma_3 = 0.2$ . The standard  $\ell_2$  and  $\ell_1$  solutions (obtained neglecting the uncertainty terms, i.e. setting  $A(\delta) = A_0$ ) result to be

$$\theta_{\text{nom}2} = \begin{bmatrix} -10 \\ -9.728 \\ 9.983 \end{bmatrix}, \quad \theta_{\text{nom}1} = \begin{bmatrix} -11.8235 \\ -11.5882 \\ 11.7647 \end{bmatrix},$$

with nominal residuals of 1.7838 and 1.8235, respectively.

Applying Theorem 1, the minimal worst-case expected  $\ell_2$  residual resulted to be  $\psi_2^* = 2.164$ , whereas the minimal upper bound on worst-case expected  $\ell_1$  residual resulted to be  $\bar{\psi}_1^* = 4.097$ . The corresponding parameter estimates are

$$\theta_{\text{ewc}2} = \begin{bmatrix} -2.3504 \\ -2.0747 \\ 2.4800 \end{bmatrix}, \quad \theta_{\text{ewc}1} = \begin{bmatrix} -2.8337 \\ -2.5252 \\ 2.9047 \end{bmatrix}.$$

We next analyzed numerically how the worst-case expected residuals increase with the level of perturbation. To this end, we consider the previous data with standard deviations on the perturbation depending on a parameter  $\rho \geq 0$ :  $\sigma_1 = \rho \cdot 0.067$ ,  $\sigma_2 = \rho \cdot 0.1$ ,  $\sigma_3 = \rho \cdot 0.2$ . A plot of the worst-case expected residuals as a function of  $\rho$  is shown in Figure 1. We observe that both  $\ell_1$  and  $\ell_2$  expected residuals tend to a constant value for large  $\rho$ .

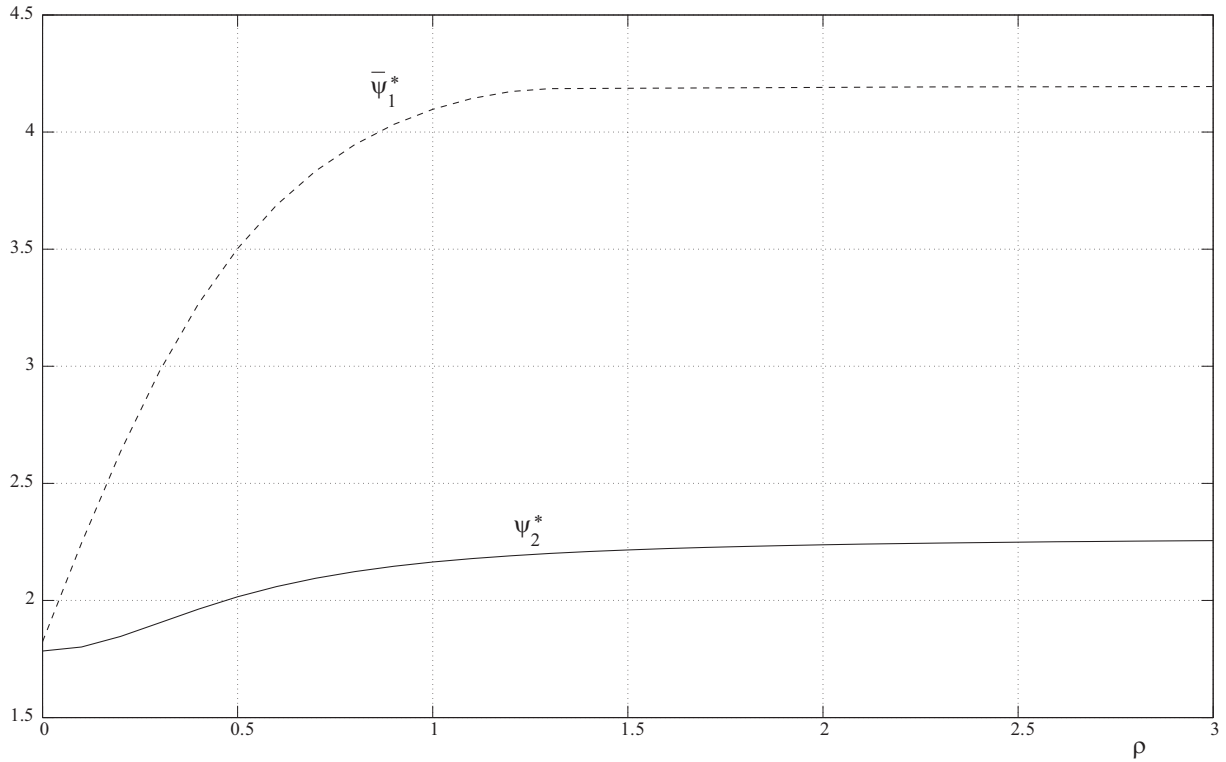


Figure 1: Plot of  $\psi_2^*$  (solid) and  $\bar{\psi}_1^*$  (dashed) as a function of perturbation level  $\rho$ .

## 5.2 Guaranteed residual at risk minimization

As a first example of guaranteed residual at risk minimization, we consider again the variable perturbation level problem of the previous section. Here, we fix the risk level to  $\epsilon = 0.1$  and solve repeatedly problems (19) and (21) for increasing values of  $\rho$ . A plot of the resulting optimal residuals at risk as a function of  $\rho$  is shown in Figure 2. These residuals grow with the covariance level  $\rho$ , as it might be expected since increasing the covariance increases the tails of the residual distribution.

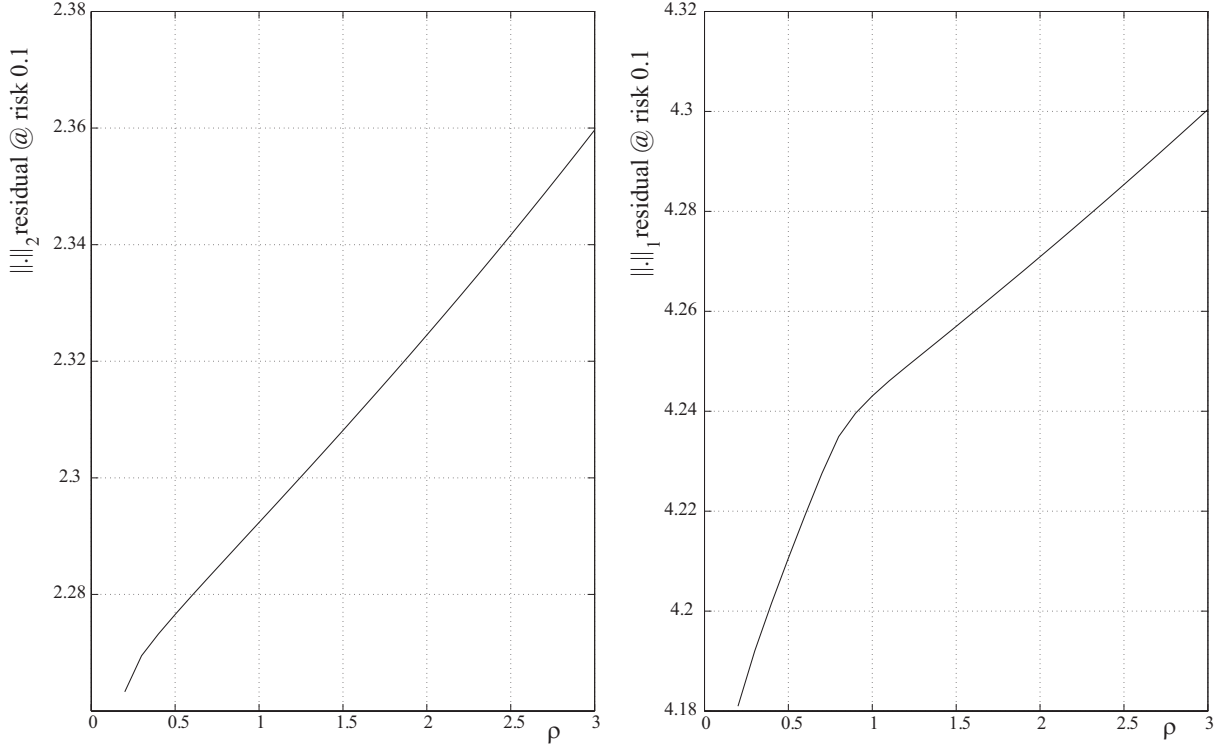


Figure 2: Worst-case  $\ell_2$  and  $\ell_1$  residuals at risk as a function of perturbation level  $\rho$ .

As a further example, we consider a system identification problem where one seeks to estimate the impulse response of a discrete-time linear FIR system from its input/output measurements. Let the system be described by the convolution

$$y_k = \sum_{j=1}^N h_j \cdot u_{k-\tau+1}, \quad k = 1, 2, \dots,$$

where  $u_i$  is the input,  $y_i$  is the output, and  $h = [h_1 \cdots h_N]$  is the impulse response to be estimated. If  $N$  input/output measurements are collected in vectors  $u^\top \doteq [u_1 \ u_2 \ \cdots \ u_N]$ ,  $y^\top \doteq [y_1 \ y_2 \ \cdots \ y_N]$ , then the impulse response  $h$  can be computed by solving the system of linear equations  $Uh = y$ , where  $U$  is a lower-triangular Toeplitz matrix having  $u$  in the first column. In practice, however, both  $u$  and  $y$  might be affected by errors, that is  $U(\delta_u) = U + \sum_{i=1}^N \delta_{ui} U_i$ ,  $y(\delta_y) = y + \sum_{i=1}^N \delta_{yi} e_i$  where  $e_i$  is the  $i$ -th column of the identity matrix in  $\mathbb{R}^N$ , and  $U_i$  is a lower-triangular Toeplitz matrix having  $e_i$  in the first column. These uncertain data are easily re-written in the form (1) by setting  $\theta = h$ ,  $q = 2N$ ,  $A_0 = U$ ,  $b_0 = y$ , and, for  $i = 1, \dots, 2N$ ,

$$A_i = \begin{cases} U_i, & \text{if } i \leq N \\ 0_N, & \text{otherwise} \end{cases}, \quad b_i = \begin{cases} 0, & \text{if } i \leq N \\ e_{i-N}, & \text{otherwise} \end{cases}, \quad \delta_i = \begin{cases} \delta_{ui}, & \text{if } i \leq N \\ \delta_{y,i-N}, & \text{otherwise} \end{cases}$$

We considered the same data used in a similar example in [8], that is  $u^\top = [1 \ 2 \ 3]$ ,  $y^\top = [4 \ 5 \ 6]$ , and assumed that the input/output measurements are affected by i.i.d. errors with zero mean and unit variance. The nominal system  $Uh = y$  has minimum-norm solution  $h_{\text{nom}} = [4 \ -3 \ 0]^\top$ . First, we solved the  $\ell_2$  residual-at-risk problem in (19), setting risk level  $\epsilon = 0.1$ . The optimal solution was achieved for  $\tau = 0.013$  and yielded an optimal worst-case residual at risk  $\gamma = 11.09$ , with corresponding parameter

$$h_{\ell_2, \text{rar}} = [0.7555 \ 0.4293 \ 0.1236]^\top.$$

This means that, no matter what the distribution on the uncertainty is (as long as it has the assumed mean and covariance), the probability of having a residual larger than  $\gamma = 11.09$  is smaller than  $\epsilon = 0.1$ .

We next performed an a-posteriori Monte-Carlo test of this solution against the nominal one  $h_{\text{nom}}$ , for two specific noise distributions, namely the uniform and the Normal distribution. The empirical cumulative distribution functions of the resulting norm residuals are shown in Figure 3. Some relevant statistics are also shown in Table 1.

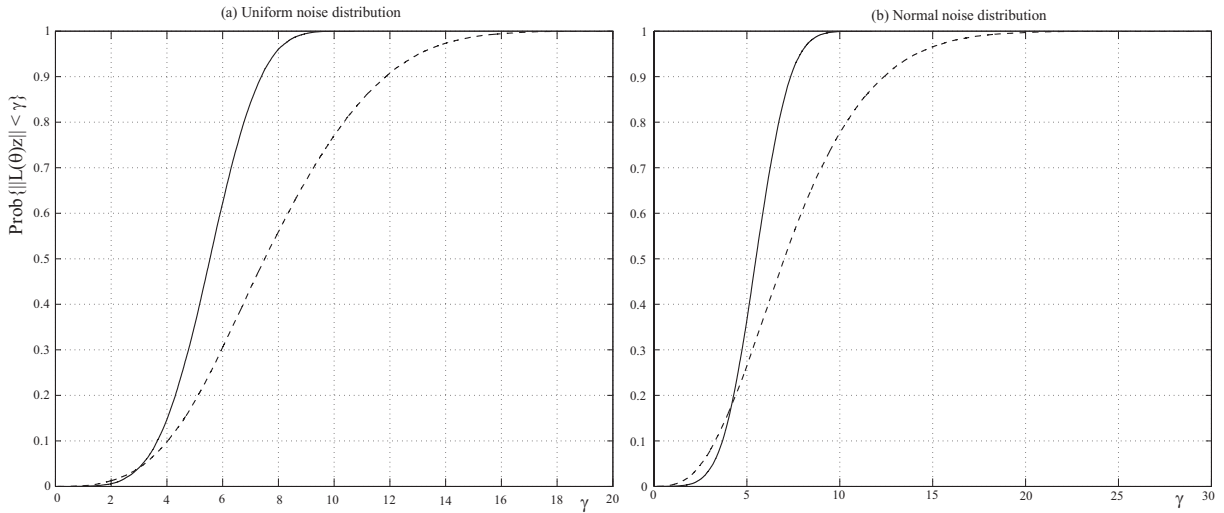


Figure 3: Empirical  $\ell_2$ -norm residual cumulative distribution for the  $h_{\ell_2, \text{rar}}$  solution (solid) and the nominal  $h_{\text{nom}}$  solution (dashed). Simulation in case (a) assumes uniform noise distribution; case (b) assumes Normal distribution.

	Uniform		Normal	
	$h_{\text{nom}}$	$h_{\ell_2, \text{rar}}$	$h_{\text{nom}}$	$h_{\ell_2, \text{rar}}$
min	0.19	0.49	0.19	0.15
max	19.40	10.79	28.98	11.75
mean	7.75	5.52	7.50	5.51
median	7.50	5.53	7.01	5.49
std	3.02	1.44	3.58	1.43
$\text{rar}@ \epsilon = 0.1$	11.85	7.39	12.39	7.36

Table 1:  $\ell_2$ -norm residual statistics from a-posteriori test on nominal and residual-at-risk solutions, with uniform and Normal distribution on the noise.

We next solved the same problem using an  $\ell_1$ -norm residual criterion. In this case, solution of problem (21) yielded  $\tau = 0.082$  and an optimal upper bound on  $\ell_1$  residual at risk

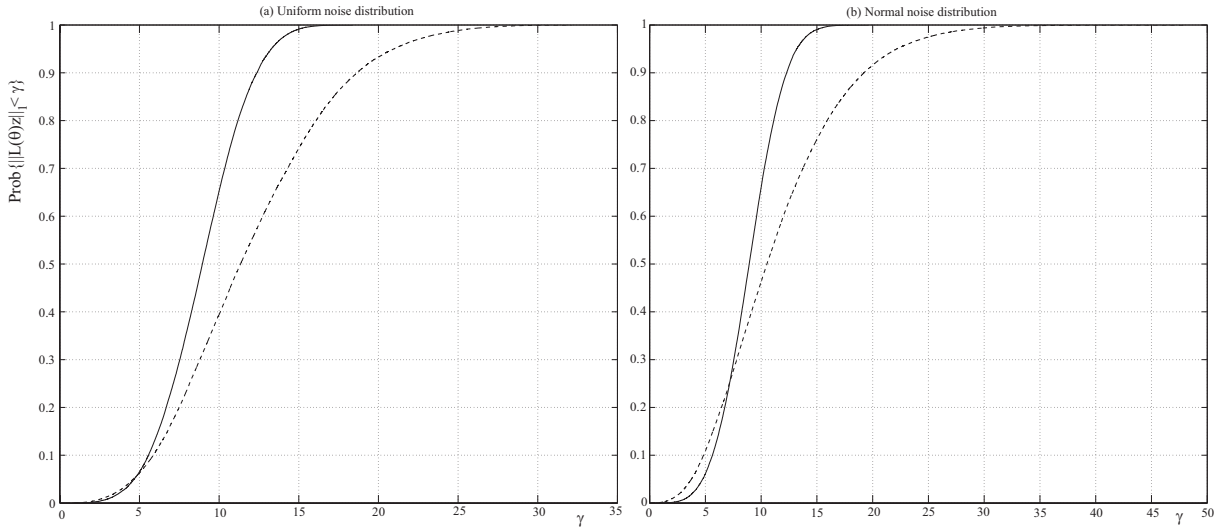


Figure 4: Empirical  $\ell_1$ -norm residual cumulative distribution for the  $h_{\ell_1, \text{rar}}$  solution (solid) and the nominal  $h_{\text{nom}}$  solution (dashed). Simulation in case (a) assumes uniform noise distribution; case (b) assumes Normal distribution.

$\gamma = 19.2$ , with corresponding parameter

$$h_{\ell_1, \text{rar}} = [0.7690 \ 0.4464 \ 0.1344]^\top.$$

An a-posteriori Monte-Carlo test then produced the residual distributions shown in Figure 4. Residual statistics are reported in Table 2.

	Uniform		Normal	
	$h_{\text{nom}}$	$h_{\ell_1, \text{rar}}$	$h_{\text{nom}}$	$h_{\ell_1, \text{rar}}$
min	0.38	0.81	0.11	0.57
max	32.12	18.39	47.93	19.63
mean	11.89	8.95	11.42	8.94
median	11.34	8.95	10.52	8.93
std	5.01	2.61	5.76	2.56
$\text{rar}@ \epsilon = 0.1$	18.66	12.37	19.22	12.29

Table 2:  $\ell_1$ -norm residual statistics from a-posteriori test on nominal and residual-at-risk solutions, with uniform and Normal distribution on the noise.

## 6 Conclusions

In this paper we discussed two criteria for linear parameter estimation in presence of random uncertain data, under both  $\ell_2$  and  $\ell_1$  norm residuals. The first criterion is a worst-case residual expectation and leads to exact and efficiently computable solutions for the  $\ell_2$  norm case. For the  $\ell_1$  norm, we can efficiently compute upper and lower bounds on the optimal solution, by means of convex second order cone programming. The second criterion considered in the paper is the worst-case residual for a given risk level  $\epsilon$ . With this criterion, an exact solution for the  $\ell_2$  norm case can be computed by a solving a sequence of

semi-definite programs, and an analogous computational effort is required for computing an upper bound on the optimal solution in the  $\ell_1$  norm case. The estimation setup proposed in the paper is “distribution free,” in the sense that only information about the mean and covariance of the random uncertainty need be available to the user: the results are guaranteed irrespective of the actual shape of uncertainty distribution.

## Appendix

### A Proof of Lemma 1

We start by recalling a preliminary result. Let  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$ , be functions whose expectations  $q_i$  are given and finite, and consider the following problem  $(P)$  and its dual  $(D)$ :

$$(P) : \quad Z^P \doteq \sup_x \mathbb{E} \{ \phi(x) \}$$

$$\text{subject to: } \mathbb{E} \{ f_i(x) \} = q_i, \quad i = 0, 1, \dots, m;$$

$$(D) : \quad Z^D \doteq \inf_y \mathbb{E} \{ y^\top f(x) \} = \inf y^\top q$$

$$\text{subject to: } y^\top f(x) \geq \phi(x), \quad \forall x \in \mathbb{R}^n,$$

where  $f_0(x) = 1$  and  $q_0 = 1$  correspond to the implied probability-mass constraint, and the infimum and the supremum are taken with respect to all probability distributions satisfying the specified moment constraints. Then, strong duality holds:  $Z^P = Z^D$ , hence  $\sup_x \mathbb{E} \{ \phi(x) \}$  can be computed by solving the dual problem  $(D)$ ; see for instance Section 16.4 of [2].

Now, the primal problem that we wish to solve in Lemma 1 for computing  $E_{\text{wc}}$  is

$$(P) : \quad Z^P = E_{\text{wc}} = \sup_x \mathbb{E} \{ \phi(x) \}$$

$$\text{subject to:} \quad \mathbb{E} \{ x \} = \hat{x}$$

$$\mathbb{E} \{ xx^\top \} = \Gamma + \hat{x}\hat{x}^\top,$$

where  $\Gamma \succ 0$  is the covariance matrix of  $x$ , and the functions  $f_i$  are  $x_k$ ,  $k = 1, \dots, n$ , and  $x_k x_j$ ,  $1 \leq k \leq j \leq n$ . Then, the dual problem is

$$(D) : \quad Z^D = \inf_{y_0 \in \mathbb{R}, y \in \mathbb{R}^n, Y = Y^\top \in \mathbb{R}^{n,n}} y_0 + y^\top \hat{x} + \text{Tr}(\Gamma + \hat{x}\hat{x}^\top)Y$$

$$\text{subject to: } y_0 + y^\top x + \text{Tr} xx^\top Y \geq \phi(x), \quad \forall x \in \mathbb{R}^n,$$

where the dual variable  $y_0$  is associated with the implicit probability-mass constraint. Defining

$$M = \begin{bmatrix} Y & \frac{1}{2}y \\ \frac{1}{2}y^\top & y_0 \end{bmatrix}, \quad Q = \begin{bmatrix} \Gamma + \hat{x}\hat{x}^\top & \hat{x} \\ \hat{x}^\top & 1 \end{bmatrix}, \quad z = \begin{bmatrix} x \\ 1 \end{bmatrix},$$

this latter problem writes

$$(D) : \quad Z^D = \inf_{M=M^\top \in \mathbb{R}^{n+1}} \text{Tr} QM$$

$$\text{subject to: } z^\top M z \geq \phi(x), \quad \forall x \in \mathbb{R}^n,$$

which is (6), thus concluding the first part of the proof.

The result in (7) can then be obtained by specializing (6) to the case when  $\phi(x) = \mathbb{I}_S(x)$ , where  $\mathbb{I}_S$  is the indicator function of set  $S$ , since  $\text{Prob}\{x \in S\} = \mathbb{E}\{\mathbb{I}_S(x)\}$ . We thus have that  $Z^P = Z^D$  for

$$\begin{aligned} (P) : \quad & Z^P = P_{\text{wc}} = \sup_x \text{Prob}\{x \in S\} \\ & \text{subject to: } \mathbb{E}\{x\} = \hat{x}, \quad \mathbb{E}\{xx^\top\} = \Gamma + \hat{x}\hat{x}^\top, \\ (D) : \quad & Z^D = \inf_{M=M^\top \in \mathbb{R}^{n+1}} \text{Tr} QM \\ & \text{subject to: } z^\top Mz \geq \mathbb{I}_S(x), \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

The constraint  $z^\top Mz \geq \mathbb{I}_S(x) \forall x \in \mathbb{R}^n$  can be rewritten as  $z^\top Mz \geq 1 \forall x \in S$ ,  $z^\top Mz \geq 0 \forall x \in \mathbb{R}^n$ , and this latter constraint is equivalent to requiring  $M \succeq 0$ , which explains (7) and concludes the proof.  $\square$

## B Matrix cube theorem

**Theorem 4 (Matrix cube relaxation; [1])** *Let  $B^0, B^1, \dots, B^L \in \mathbf{R}^{n \times n}$  be symmetric and  $B^1, \dots, B^L$  be of rank two. Let the problem  $P_\rho$  be defined as:*

$$P_\rho : \text{Is } B^0 + \sum_{i=1}^L u_i B^i \succeq 0, \quad \forall u : \|u\|_\infty \leq \rho? \quad (23)$$

and the problem  $P_{\text{relax}}$  be defined as:

$P_{\text{relax}}$ : *Do there exist symmetric matrices  $X_1, \dots, X_L \in \mathbf{R}^{n \times n}$  satisfying*

$$\begin{aligned} X_i &\succeq \pm \rho B^i, \quad i = 1, \dots, L, \\ \sum_{i=1}^L X_i &\preceq B^0? \end{aligned}$$

Then, the following statements hold:

1. *If  $P_{\text{relax}}$  is feasible, then  $P_\rho$  is feasible.*
2. *If  $P_{\text{relax}}$  is not feasible, then  $P_{\frac{\pi}{2}\rho}$  is not feasible.*  $\triangleleft$

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