V. Conclusion

In this paper some conditions under which, even in the uncertain case, the convergence to a sliding manifold can be attained relying on a control strategy still based on a simplex of control vectors are identified. The contribution is expressed through a couple of theorems which appear as a natural basis to develop a multi-input control strategy which operating a sequence of rotations and increments of the modula of the vectors of the known simplex enables one to attain the condition relevant to the reciprocal position of the uncertain and the known simplexes expressed by the Theorem 4 statement. Thus, even in the uncertain case, the objective of steering y, and consequently s, to the origin of their state spaces is reached.

References

[1] S. V. Baida and D. B. Izosimov, "Vector method of design of sliding motion and simplex algorithms," *Automation Remote Contr.*, vol. 46, pp. 830–837, 1985.

[2] G. Bartolini and T. Zolezzi, "Control of nonlinear variable structure systems," *J. Math. Analysis Appl.*, vol. 118, pp. 42–62, 1986.

[3] ——, "Behavior of variable structure control systems near the sliding manifold," *Syst. Contr. Lett.*, vol. 21, pp. 43–48, 1993.

[4] B. E. Paden and S. S. Sastry, "A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 73–82, 1987.

[5] V. I. Utkin, *Sliding Modes and Their Application in Variable Structure Systems.* Moscow: MIR, 1978.

[6] ——, *Sliding Modes in Control and Optimization.* Berlin: Springer-Verlag, 1992.

[7] A. S. I. Zinober, Ed., *Deterministic Control of Uncertain Systems.* London: Peregrinus, 1990.

[8] ——, *Variable Structure and Lyapunov Control.* London: Springer-Verlag, 1994.

LMI Optimization for Nonstandard Riccati Equations Arising in Stochastic Control

Mustapha Ait Rami and Laurent El Ghaoui

***Abstract*— We consider coupled Riccati equations that arise in the optimal control of jump linear systems. We show how to reliably solve these equations using convex optimization over linear matrix inequalities (LMI's). The results extend to other nonstandard Riccati equations that arise, e.g., in the optimal control of linear systems subject to state-dependent multiplicative noise. Some nonstandard Riccati equations (such as those connected to linear systems subject to both state- and control-dependent multiplicative noise) are not amenable to the method. We show that we can still use LMI optimization to compute the optimal control law for the underlying control problem without solving the Riccati equation.**

Notation

For a real matrix A, $A > 0$ (respectively, $A \geq 0$) means A is symmetric and positive definite (respectively, positive semidefinite). $\mathcal{S}_n$ denotes the set of real symmetric matrices of order n.

Manuscript received August 3, 1995; revised February 3, 1996. This work was supported in part by DRET under Contract 92017-BC14.

The authors are with the Ecole Nationale Supérieure de Techniques Avancées, 75739 Paris, France (e-mail: elghaoui@ensta.fr).

Publisher Item Identifier S 0018-9286(96)07509-5.

I. Introduction

We consider the problem of finding positive semidefinite solutions to the following set of coupled Riccati equations:

$$\begin{aligned}\mathcal{R}_i(X_1, \cdots, X_N) &\triangleq A_i^T X_i + X_i A_i - X_i B_i R_i^{-1} B_i^T X_i \\ &\quad + \sum_{j=1}^{N} \pi_{ij} X_j + Q_i = 0, \\ &\qquad i = 1, \cdots, N. \quad (1)\end{aligned}$$

In the above, A_i, B_i, Q_i, and R_i are real matrices of dimensions $n \times n$, $n \times n_u$, $n \times n$, and $n_u \times n_u$, respectively. For every i, $i = 1, \cdots, N$, Q_i is symmetric positive semidefinite and R_i is symmetric positive definite.

Equation (1) arises in the optimal control of jump linear systems defined by

$$\frac{dx}{dt} = A(r(t))x + B(r(t))u \tag{2}$$

where $A(r(t)) = A_i$, $B(r(t)) = B_i$ when $r(t) = i$. The process r: $\mathbf{R}^+ \to \{1, \cdots, N\}$ is Markovian with transition probabilities given by

$$\mathbf{Prob}\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases}\pi_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta + o(\Delta) & \text{else.}\end{cases}$$

Here $\pi_{ij} \geq 0$ for $i \neq j$ and $-\pi_{ii} = \sum_{i \neq j} \pi_{ij}$.

In the so-called jump linear-quadratic (JLQ) problem, one seeks to minimize the quadratic cost

$$J(u) \triangleq \mathbf{E}\left\{\int_0^\infty (x^T Q(r(t))x + u^T R(r(t))u)dt \mid x(0),\ r(0)\right\} \tag{3}$$

subject to (2). The cost matrices $Q(r(t))$ and $R(r(t))$ are defined by

$$Q(r(t)) = Q_i, \quad R(r(t)) = R_i \quad \text{when } r(t) = i.$$

Under certain assumptions detailed below, Ji and Chizeck [9] have shown that there exists a unique stabilizing optimal control law. This control law is given by

$$u(t) = K_i x(t) \quad \text{when } r(t) = i \tag{4}$$

where $K_i = -R_i^{-1} B_i S_i$, and S_i, $i = 1, \cdots, N$, are the (unique) positive-definite solutions of the set of coupled Riccati equations $\mathcal{R}_i(S_1, \cdots, S_N) = 0$, $i = 1, \cdots, N$.

Thus, solving the optimal control problem (3) amounts to solving the nonstandard Riccati (1). As opposed to the deterministic case, there is no theory which connects the solutions of the equation to the eigenvectors of a Hamiltonian matrix [10], [15]. This is due to the coupling between the variables via the transition probability rates π_{ij}. As a consequence, an alternate numerical procedure has to be used.

Several numerical methods have been proposed for solving the problem. Wonham [18] proposed a quasilinearization method, reminiscent of early methods used for standard Riccati equations. To prove convergence of this scheme a hypothesis is required, and this hypothesis is only sufficient and difficult to check. Later, Mariton and Bertrand [13] proposed a homotopy algorithm for solving the problem. Recently, Abou-Khandil *et al.* [1] proposed two numerical methods based on the solution of uncoupled Riccati equations. For initializing these algorithms (and also, proving convergence), it is required to find a solution of related coupled Riccati inequalities which in some cases is difficult, or even impossible (see Section VI for a numerical example). For the discrete-time equivalent to this problem, we mention the dynamic programming approach of Chizeck *et al.* [7] and the algorithm of [2]. Also, note that the paper by Yan *et al.* [19] deals with a related (but different) nonlinear matrix equation.

In this paper, we present a method to solve (1). The method is based on convex optimization over linear matrix inequalities (LMI's). A consequence of our formulation is that the problem can be solved in polynomial time [14], using currently available software [16], [6].

We show that provided the system is stabilizable (in the mean-square sense), our algorithm always yields the "maximal" positive semidefinite solution to the Riccati equation (1). If we further assume that each mode $(Q_i^{1/2}, A_i)$ is observable, the results of [9] imply that mean-square stabilizability is necessary and sufficient for the optimal control problem (3) to have a stabilizing solution. In this case, our algorithm provides this (unique) optimal control law.

Important features of our approach are the following. The hypothesis required for our algorithm to work (the mean-square stabilizability condition) is a) natural from a control point of view and b) reliably checked using LMI optimization. Finally, a similar algorithm can be applied to other nonstandard Riccati equations, such as those that arise in $\mathbf{H}_\infty$-optimal control (see [3]).

The paper is organized as follows. We define the notion of mean-square stability in Section II. We give our main result in Section III. In Section IV, we list some other optimal control problems that can be solved using the approach, including discrete-time systems and $\mathbf{H}_\infty$ state-feedback control for jump linear systems. In Section V, we consider a problem where the proposed algorithm approach cannot be applied to solve the Riccati equation. We show that the corresponding optimal control problem can still be solved using LMI optimization (that is, we are able to solve for the optimal control law without solving the Riccati equation). In Section VI, we provide three numerical examples. The first is taken from [1]. The second example shows that sometimes the hypothesis required in [1] for initializing the algorithm is difficult to check. We show that our algorithm behaves equally well in this case. The last example illustrates the results in Section V. All our proofs are given in the Appendix.

II. Preliminaries

Definition 2.1: System (2) is mean-square stabilizable if there exists a control law of the form (4) such that the closed-loop system is stable in the mean-square sense, that is, if for every initial condition $x(0)$

$$\lim_{t\to\infty} \mathbf{E}x(t)x(t)^T = 0.$$

In this paper, we make the following hypothesis:

H) System (2) is mean-square stabilizable.

The following theorem shows in particular that mean-square stabilizability can be reliably checked via LMI optimization. (For a proof, see [5].)

Theorem 2.1: The following properties are equivalent.

1) System (2) is mean-square stabilizable.
2) There exist $K_1, \cdots, K_N$ and $P_1, \cdots, P_N > 0$ such that

$$(A_i + B_iK_i)^T P_i + P_i(A_i + B_iK_i) + \sum_{j=1}^N \pi_{ij}P_j < 0.$$

3) There exist $K_1, \cdots, K_N$ such that, for all matrices $T_1, \cdots, T_N$, there exist unique matrices $S_1, \cdots, S_N$ such that

$$(A_i + B_iK_i)^T S_i + S_i(A_i + B_iK_i) + \sum_{j=1}^N \pi_{ij}S_j + T_i = 0.$$

If, for every i, $T_i > 0$ (respectively, $T_i \geq 0$) then $S_i > 0$ (respectively, $S_i \geq 0$).
4) There exist $Y_1, \cdots, Y_N$ and $Q_1, \cdots, Q_N$ such that the following LMI holds:

$$A_iQ_i + B_iY_i + Q_iA_i^T + Y_i^TB_i^T + \sum_{j=1}^N \pi_{ji}Q_j < 0,$$
$$Q_i = Q_i^T > 0, \quad i = 1, \cdots, N. \quad (5)$$

Remark: The above theorem yields a numerically efficient way of checking mean-square stabilizability using convex optimization over LMI's [16], [6]. If the LMI (5) holds, then a stabilizing control law is given by (4) with $K_i = Y_iQ_i^{-1}$.

III. Main Result

Consider the following optimization problem:

maximize $\mathbf{Tr}\, X_1 + \cdots + \mathbf{Tr}\, X_N$

$$\text{subject to} \begin{bmatrix} A_i^TX_i + X_iA_i + \sum_{j=1}^N \pi_{ij}X_j + Q_i & X_iB_i \\ B_i^TX_i & R_i \end{bmatrix} \geq 0,$$
$$X_i = X_i^T, \quad i = 1, \cdots, N. \quad (6)$$

We define the optimal set, and denote by $\mathcal{P}^{\text{opt}}$, the set N-tuples $(X_1, \cdots, X_N)$ of maximizers of problem (6). (We note that $\mathcal{P}^{\text{opt}}$ is not empty, since the constraints are always feasible.)

Theorem 3.1: If hypothesis **H**) holds, then we have the following.

- The optimal set is a singleton, $\mathcal{P}^{\text{opt}} = \{(P_1^{\text{opt}}, \cdots, P_N^{\text{opt}})\}$.
- For every i, $P_i^{\text{opt}} \geq 0$.
- For every i, $\mathcal{R}_i(P_1^{\text{opt}}, \cdots, P_N^{\text{opt}}) = 0$.
- The solution is maximal. That is, for every symmetric matrix $X_1, \cdots, X_N$ such that $\mathcal{R}_i(X_1, \cdots, X_N) \geq 0$, $i = 1, \cdots, N$, we have $P_i^{\text{opt}} \geq X_i$.

The theorem shows that by solving an LMI problem, we obtain a (unique) optimal solution of (3), provided **H**) holds.

It can be shown that if **H**) holds, then the control law

$$K_i^{\text{opt}} = -R_i^{-1}B_i^TP_i^{\text{opt}}, \quad i = 1, \cdots, N \quad (7)$$

minimizes the quadratic cost (3). However, this solution is not necessary stabilizing. The following result is due to Ji and Chizeck [9].

Theorem 3.2: If hypothesis **H**) holds, and if in addition, each mode $(Q_i^{1/2}, A_i)$ is observable, then the optimal control law given by (7) is stabilizing in the mean-square sense.

As seen from the purely deterministic case ($\pi_{ij} = 0$), the above observability hypothesis of each mode is not necessary for the maximal solution to yield a stabilizing control law.

IV. Other Nonstandard Riccati Equations

Our result, and the proof given in the Appendix, can be easily extended to other types of nonstandard Riccati equations arising in the optimal control of several classes of linear systems. These other problems include the following.

Discrete-Time Jump Linear Systems: The discrete-time jump linear systems are described by (see [7] and [8])

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k$$

where $A(r_k) = A_i, B(r_k) = B_i$ when $r_k = i$, $i = 1, \cdots, N$. The process r_k is a finite-state discrete-time Markov chain with transition probabilities

$$\mathbf{Prob}\{r_{k+1} = j \mid r_k = i\} = p_{ij}, \quad 1 \leq i, j \leq N.$$

In the optimal control of this type of system, one obtains a Riccati equation of the form

$$A_i^T S_i A_i - X_i - A_i^T B_i S_i \left[R_i + B_i^T S_i B_i\right]^{-1} B_i^T S_i A_i + Q_i = 0$$
$$S_i = \sum_{j=1}^{N} p_{ij} X_j, \quad X_i \geq 0, \quad i = 1, \cdots, N \tag{8}$$

where $Q_i \geq 0$, $R_i > 0$ are the state and control weighting matrices, respectively.

Linear Systems with Both Jumps and Multiplicative Noise on the State: These systems are described in [12]. The corresponding Riccati equations have the form

$$A_i^T X_i + X_i A_i - X_i B_i R_i^{-1} B_i^T X_i + \sum_{j=1}^{N} \pi_{ij} X_j + \Delta_i(X_i) + Q_i = 0, \quad i = 1, \cdots, N \tag{9}$$

where Δ_i's are some linear matrix functions from $\mathcal{S}_n$ to $\mathcal{S}_n$ that are positive (that is, $X \geq 0$ implies $\Delta_i(X) \geq 0$).

Linear Systems with Both Jumps and Random State Discontinuities: These systems are described in [11]. In this case, the Riccati equations have the form

$$A_i^T X_i + X_i A_i - X_i B_i R_i^{-1} B_i^T X_i + \sum_{j=1}^{N} \pi_{ij}(X_j + \Gamma_{ij}(X_j)) + Q_i = 0, \quad i = 1, \cdots, N \tag{10}$$

where the Γ_{ij}'s are linear matrix functions from $\mathcal{S}_n$ to $\mathcal{S}_n$ that are positive.

For all the above kinds of nonstandard Riccati equations, the maximal solution can be obtained by maximizing $\mathbf{Tr}(X_1 + \cdots + X_N)$ subject to the corresponding Riccati inequalities (in which the equality sign is replaced by a $\geq$ sign).

$\mathbf{H}_\infty$*-Optimal Control of Jump Linear Systems:* In [3], de Souza and Fragoso have shown that a class of $\mathbf{H}_\infty$-optimal control problems for jump linear systems could be solved via a set of coupled Riccati equations of the form

$$A_i^T X_i + X_i A_i + X_i \left[\frac{1}{\gamma^2} B_{1i} B_{1i}^T - B_{2i} B_{2i}^T\right] X_i + \sum_{j=1}^{N} \pi_{ij} X_j + C_i^T C_i = 0, \quad i = 1, \cdots, N \tag{11}$$

where $\gamma > 0$ is a prescribed level of disturbance attenuation. We assume that the system is mean-square stabilizable and (C_i, A_i) is observable for all $i = 1, \cdots, N$. The mean-square stabilizing solution can be computed by *minimizing* γ^2 subject to

$$A_i^T X_i + X_i A_i + X_i \left[\frac{1}{\gamma^2} B_{1i} B_{1i}^T - B_{2i} B_{2i}^T\right] X_i + \sum_{j=1}^{N} \pi_{ij} X_j + C_i^T C_i \leq 0, \quad X_i > 0, \quad i = 1, \cdots, N. \tag{12}$$

The above constraints are easily transformed into a LMI's by introducing variables $Y_i = X_i^{-1}$ and rewriting (12) as

$$A_i Y_i + Y_i A_i^T + \pi_{ii} Y_i + \frac{1}{\gamma^2} B_{1i} B_{1i}^T - B_{2i} B_{2i}^T + \sum_{j \neq i}^{N} \pi_{ij} Y_i Y_j^{-1} Y_i + Y_i C_i^T C_i Y_i \leq 0, \quad Y_i > 0, \quad i = 1, \cdots, N.$$

The above constraint can be expressed by the following LMI's:

$$\begin{bmatrix} L_i & M_i \\ M_i^T & D_i \end{bmatrix} \leq 0, \quad i = 1, \cdots, N$$

where

$$L_i = \begin{bmatrix} A_i Y_i + Y_i A_i^T + \pi_{ii} Y_i + \frac{1}{\gamma^2} B_{1i} B_{1i}^T - B_{2i} B_{2i}^T & Y_i C_i^T \\ C_i Y_i & -I \end{bmatrix}$$
$$M_i = \left[\sqrt{\pi_{i1}} Y_i \ \cdots \ \sqrt{\pi_{i(i-1)}} Y_i \ \sqrt{\pi_{i(i+1)}} Y_i \ \cdots \ \sqrt{\pi_{iN}} Y_i\right]$$
$$D_i = -\mathbf{diag}(Y_1, \cdots, Y_{i-1}, Y_{i+1}, \cdots, Y_N), \quad i = 1, \cdots, N.$$

V. Problems not Amenable to the Method

Some nonstandard Riccati equations arising in optimal control of stochastic systems cannot be solved using the proposed approach. The corresponding optimal control problem (that is, the optimal state-feedback control law) can still be computed using LMI optimization.

One instance of such a problem is the following. Consider a system with multiplicative noise on both state and input matrices [18]. The system satisfies the Itô differential equation

$$dx(t) = Ax(t)dt + Bu(t)dt + \sum_{i=1}^{L} (A_i x(t) + B_i u(t)) dp_i(t) \tag{13}$$

where u is the command input and p_i are independent, Brownian motions with variance σ_i, $i = 1, \cdots, L$.

The optimal control problem under consideration is to minimize

$$J(u) \triangleq \mathbf{E}\left\{\int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t)) dt \mid x(0)\right\} \tag{14}$$

where $Q \geq 0$, $R > 0$ are the state and control weighting matrices, respectively.

The Riccati equation associated with the optimal control problem is [18]

$$A^T S + SA - SB\left(R + \sum_{i=1}^{L} \sigma_i^2 B_i^T S B_i\right)^{-1} B^T S + \sum_{i=1}^{L} \sigma_i^2 A_i^T S A_i + Q = 0. \tag{15}$$

In the next section, we provide an example showing that the solution of such equations might not be obtained using our approach (that is, by maximizing $\mathbf{Tr}S$ subject to (15), where equality sign is replaced with $\geq$ sign). This is due to the fact that there is no maximal solution in this case. (As we pointed out before, the approach can be proven to work when there is no control-dependent noise, $B_i = 0$, $i = 1, \cdots, L$).

Although we are not able to solve the Riccati equation, we can still solve for the optimal control law using LMI optimization via a stochastic Lyapunov function approach (see [4]). Indeed, the optimal control law is given by $u = UX^{-1}x$, where U, X are solutions to the following LMI problem:

$$\begin{gathered} \text{Minimize } \mathbf{Tr}(Y) + \mathbf{Tr}(XQ) \text{ subject to} \\ \begin{bmatrix} Y & R^{1/2} U \\ U^T R^{1/2} & X \end{bmatrix} > 0, \\ AX + BU + XA^T + U^T B^T + x(0)x(0)^T \\ + \sum_{i=1}^{L} \sigma_i^2 (A_i X + B_i U) X^{-1} (A_i X + B_i U)^T < 0. \end{gathered} \tag{16}$$

(Note that, as in the deterministic case ($\sigma_i = 0$), the optimal control law does not depend on the choice of $x(0) \neq 0$.)

To sum up, we have now two LMI-based methods for solving optimal control problems for stochastic linear systems. The first is based on Theorem 3.1 and does not apply to some problems. Moreover, the corresponding control law [as given by (7)] is not necessarily stabilizing. The second method consists of solving the optimal control problem "directly" (without solving the Riccati equation). The second method is more demanding numerically (the LMI problem has more variables and involves bigger matrices), but the resulting control law is always stabilizing.

VI. Numerical Experiments

We first consider the example given in [1] consisting of a third-order system having three modes

$$A_1 = \begin{bmatrix} -2.5 & 0.3 & 0.8 \\ 1 & -3 & 0.2 \\ 0 & 0.5 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2.5 & 1.2 & 0.3 \\ -0.5 & 5 & -1 \\ 0.25 & 1.2 & 5 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 2 & 1.5 & -0.4 \\ 2.2 & 3 & 0.7 \\ 1.1 & 0.9 & -2 \end{bmatrix}$$
$$B_1 = B_2 = B_3 = \mathbf{diag}(0.707 \quad 1 \quad 1)$$
$$\Pi = \begin{bmatrix} -3 & 0.5 & 2.5 \\ 1 & -2 & 1 \\ 0.7 & 0.3 & -1 \end{bmatrix}, \quad Q_1 = \mathbf{diag}(25 \quad 1 \quad 11)$$
$$Q_2 = \mathbf{diag}(37 \quad 70 \quad 34), \qquad Q_3 = \mathbf{diag}(10 \quad 16 \quad 21)$$
$$R_1 = R_2 = R_3 = I.$$

Solving problem (6) directly yields

$$P_1^{\rm opt} = \begin{bmatrix} 5.0203 & 1.0087 & 0.3839 \\ 1.0087 & 2.5138 & 0.3520 \\ 0.3839 & 0.3520 & 3.0135 \end{bmatrix}$$
$$P_2^{\rm opt} = \begin{bmatrix} 5.3256 & 0.2985 & 0.5192 \\ 0.2985 & 13.9462 & 0.7286 \\ 0.5192 & 0.7286 & 19.8938 \end{bmatrix}$$
$$P_3^{\rm opt} = \begin{bmatrix} 9.6982 & 3.7658 & 0.2102 \\ 3.7658 & 8.2582 & 0.6635 \\ 0.2102 & 0.6635 & 3.4688 \end{bmatrix}.$$

The corresponding residual is

$$\|\mathcal{R}_1\| = 2.2 \times 10^{-9}, \quad \|\mathcal{R}_2\| = 3.4 \times 10^{-9}$$
$$\|\mathcal{R}_3\| = 1.78 \times 10^{-9}.$$

The solution took about 4×10^4 flops and 9 seconds on an HP-710 workstation using the general-purpose LMI solver **SP** [16] and its Matlab interface LMITOOL [6]. In this example, there is no feasibility phase, since $Q_i > 0$ implies that $X_i = 0$ is strictly feasible. It took 23 iterations for **SP** to find the optimal point. We have used a very small tolerance for the convergence test (see [16]): the parameters abstol and reltol were both set to 10^{-10}.

The second example is one where the search for an initial guess for the algorithm of [1] is not trivial. Consider

$$A_1 = \begin{bmatrix} 1.5 & 1.0 \\ 0 & -1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.5 & 0 \\ 1.0 & 1.5 \end{bmatrix}$$
$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1.0 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} -1.0 & 1.0 \\ 1.0 & -1.0 \end{bmatrix}$$
$$Q_1 = Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_1 = R_2 = 1.$$

The above system is mean-square stabilizable. To prove it, we have found a feasible point to the LMI's (5). One such point is

$$Q_1 = \begin{bmatrix} 4.7592 & -7.8929 \\ -7.8929 & 21.4985 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 6.266 & 0 \\ 0 & 9.6185 \end{bmatrix}$$
$$Y_1 = [-29 \quad -37\,168], \quad Y_2 = [2 \quad -37\,227].$$

In this case, finding an initial point for the algorithm of [1] is not trivial. We have to find a positive semidefinite solution to the *nonconvex* matrix inequalities

$$\left(A_i + \frac{\pi_{ii}}{2} I\right)^T K_i + K_i \left(A_i + \frac{\pi_{ii}}{2} I\right) + Q_i - K_i B_i R_i^{-1} B_i^T K_i + \sum_{j \neq i} \pi_{ij} K_j \leq 0, \quad i = 1, \cdots, N. \tag{17}$$

In [1, Remark 3-ii)], it is recommended to take, whenever possible, $K_i = \alpha I$, where $\alpha > 0$ is large enough. It turns out that no such α exist in this case. The difficulty here is due to the fact that the quadratic term $K_i B_i R_i^{-1} B_i^T K_i$ might be rank-deficient. (We note that finding a solution to (17), or proving there is none, can be formulated as an LMI feasibility problem in K_i^{-1}.)

Solving problem (6) directly yields (in 6012 flops and 3 seconds)

$$P_1^{\rm opt} = \begin{bmatrix} 30.4839 & 8.3271 \\ 8.3271 & 2.9888 \end{bmatrix}, \quad P_2^{\rm opt} = \begin{bmatrix} 7.3721 & 2.7307 \\ 2.7307 & 3.2336 \end{bmatrix}.$$

The corresponding residual is $\|\mathcal{R}_1\| = 5.5 \times 10^{-9}$, $\|\mathcal{R}_2\| = 4.3 \times 10^{-10}$.

Finally, we give an example showing that the approach might not give a solution for Riccati equations of the type (15). Consider (13) with

$$A = \begin{bmatrix} 0.9089 & 0.8609 \\ 0.2501 & 0.4713 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5060 \\ 0.6004 \end{bmatrix}$$
$$[\sigma_1 \quad \sigma_2] = [0.8766 \quad 0.4400]$$
$$[A_1 \quad A_2] = \begin{bmatrix} 0.8176 & 0.4622 & 0.6327 & 0.8247 \\ 0.7558 & 0.9514 & 0.4393 & 0.6890 \end{bmatrix}$$
$$[B_1 \quad B_2] = \begin{bmatrix} 0.2893 & 0.5144 \\ 0.5374 & 0.1034 \end{bmatrix}$$

and with the cost (14), with $Q = I$, $R = I$. (As said in Section V, the value of the initial condition is arbitrary as long as it is not zero.)

The above system is mean-square stabilizable in the sense of the definition in Section II. To prove this claim, one has to check whether the inequalities

$$AQ + QA^T + BY + Y^T B^T + \sum_{i=1}^{L} \sigma_i^2 (A_i Q + B_i Y) Q^{-1} (A_i Q + B_i Y)^T < 0$$

have a solution $Q > 0$, Y (see [4]). This is an LMI feasibility problem which we have solved using LMITOOL. We have obtained a feasible solution as

$$Q = \begin{bmatrix} 8.1580 & 3.6020 \\ 3.6020 & 33.1355 \end{bmatrix}, \quad Y = [-31.8056 \quad -51.5768].$$

However, when we solved the optimization problem

$$\text{maximize } \mathbf{Tr}\, P \text{ subject to (15)}$$
$$\text{(with equality sign replaced with } \geq \text{ sign)}$$

we obtained

$$P^{\rm opt} = \begin{bmatrix} 3930.8 & -1775.3 \\ -1775.3 & 3387.5 \end{bmatrix}.$$

The corresponding residual is 7.853×10^3. This shows that in this example, there exists no maximal solution to the equation. (Note that when we set $B_i = 0$, we do obtain a solution

$$P^{\rm opt} = \begin{bmatrix} 20.3843 & -5.9345 \\ -5.9345 & 9.8843 \end{bmatrix}$$

with residual 5.4318×10^{-9}.)

Although we were not able to solve the Riccati equation in the case $B_i \neq 0$, we could solve for the optimal control law by solving (16). This resulted in the control law

$$K = UX^{-1} = [-3.2130 \quad -1.7638].$$

VII. Conclusion

We have devised a reliable method for solving some nonstandard Riccati equation arising in the optimal control of various stochastic systems. The method is based on convex optimization only. This avoids problems of convergence and/or initial guess search. In fact, this formulation shows that the problem considered is *tractable*, both theoretically (solvable in polynomial time) and practically. General-purpose codes are now available to solve LMI problems. With no doubt, the computational work needed to solve the problem could be greatly improved using a special code taking into account the structure of the LMI problem at hand (see [17] for a discussion on this topic).

In some cases (as for linear systems subject to both state- and control-dependent multiplicative noise), the proposed approach breaks down. Although we cannot solve the corresponding Riccati equation, we can still compute the optimal state-feedback law using LMI optimization. This alternate method is more demanding numerically. Its advantage is to be very general and to always provide a stabilizing control law whenever the system is mean-square stabilizable.

Appendix
Proof of Theorem 3.1

We begin by proving that under hypothesis **H)**, the coupled Riccati equations in (1) have a "maximal" solution, as defined in Theorem 3.1. This will immediately imply Theorem 3.1.

Case When $Q_i > 0$, $i = 1, \cdots, N$: Since $\mathcal{R}_i(0, \cdots, 0) = Q_i > 0$, the set of symmetric matrices satisfying $\mathcal{R}_i(X_1, \cdots, X_N) \geq 0$ is not empty. By **H)**, (2) is mean-square stabilizable. Thus, by the result of Theorem 2.1, there exist $K(0)_i$ and $X(0)_i$ $(i = 1, \cdots, N)$ such that the matrices $A(0)_i = A_i - B_i K(0)_i$ satisfy

$$\begin{aligned} A(0)_i^T X_i(0) + X_i(0) A_i(0) + \sum_{j=1}^N \pi_{ij} X_j(0) \\ = -K_i(0)^T R_i K_i(0) - Q_i, \quad i = 1, \cdots, N. \end{aligned}$$

Let $\hat{K}_i(0) = K_i(0) - R_i^{-1} B_i^T \tilde{X}_i$, where $\tilde{X}_1, \cdots, \tilde{X}_N$ is any solution to the inequalities $\mathcal{R}_i(\tilde{X}_1, \cdots, \tilde{X}_N) \geq 0$. We have

$$\begin{aligned} &A_i(0)^T (X_i(0) - \tilde{X}_i) + (X_i(0) - \tilde{X}_i) A_i(0) \\ &+ \sum_{j=1}^N \pi_{ij} (X_j(0) - \tilde{X}_j) \\ &= -\hat{K}_i(0)^T R_i \hat{K}_i(0) - \mathcal{R}_i(\tilde{X}_1, \cdots, \tilde{X}_N) \leq 0. \end{aligned}$$

The result of Theorem 2.1 then implies that $X_i(0) \geq \tilde{X}_i$, for $i = 1, \cdots, N$.

Suppose that we can define sequences $K_1(m), \cdots, K_N(m)$, $A_1(m), \cdots, A_N(m)$, and $X_1(m), \cdots, X_N(m)$ by the following recursion:

$$\begin{gathered} X_i(0) \geq X_i(1) \geq \cdots \geq X_i(l) \geq \tilde{X}_i \\ A_i(m) = A_i - B_i K_i(m); \quad K_i(m) = R_i^{-1} B_i^T X_i(m-1) \\ A_i(m)^T X_i(m) + X_i(m) A_i(m) + \textstyle\sum_{j=1}^N \pi_{ij} X_j(m) \\ = -K_i(m)^T R_i K_i(m) - Q_i \\ i = 1, \cdots, N, \quad m = 1, \cdots, l. \end{gathered} \tag{18}$$

Using the fact that $Q_i > 0$ and Theorem 2.1, we have that $X_i(m) > 0$ for every m, $m = 1, \cdots, l$.

Suppose that the matrices $(A_i(0), K_i(0), X_i(0)), \cdots, (A_i(l), K_i(l), X_i(l))$ are defined as before. We now show that we can define $A_i(l+1), K_i(l+1), X_i(l+1)$ with the above recursion.

Define $K_i(l+1) = R_i^{-1} B_i^T X_i(l)$ and $A_i(l+1) = A_i - B_i K_i(l+1)$ which satisfy the following equalities:

$$\begin{aligned} &A_i(l+1)^T X_i(l) + X_i(l) A_i(l+1) + \sum_{j=1}^N \pi_{ij} X_j(l) \\ &= -(K_i(l+1) - K_i(l))^T R_i (K_i(l+1) - K_i(l)) \\ &\quad - K_i(l+1)^T R_i K_i(l+1) - Q_i. \end{aligned}$$

Recalling that $X_i(l) > 0$, we can apply Theorem 2.1 to conclude that (2), with the feedback control law with gains $K_i(l+1)$, is mean-square stabilizable. Thus, there exist $X_i(l+1)$ solutions of

$$\begin{aligned} &A_i(l+1)^T X_i(l+1) + X_i(l+1) A_i(l+1) \\ &+ \sum_{j=1}^N \pi_{ij} X_j(l+1) = -K_i(l+1)^T R_i K_i(l+1) - Q_i. \end{aligned}$$

Now define $\hat{K}_i(l+1) = K_i(l+1) - R_i^{-1} B^T \tilde{X}_i$. We obtain that

$$\begin{aligned} &A_i(l+1)^T (X_i(l+1) - \tilde{X}_i) + (X_i(l+1) - \tilde{X}_i) A_i(l+1) \\ &= -\sum_{j=1}^N \pi_{ij} (X_j(l+1) - \tilde{X}_j) - \mathcal{R}_i(\tilde{X}_1, \cdots, \tilde{X}_N) \\ &\quad - \hat{K}_i(l+1)^T R_i \hat{K}_i(l+1) \leq 0. \end{aligned}$$

Using (18) we also have

$$\begin{aligned} &A_i(l+1)^T (X_i(l) - X_i(l+1)) + (X_i(l) - X_i(l+1)) A_i(l+1) \\ &= -\sum_{j=1}^N \pi_{ij} (X_j(l) - X_j(l+1)) \\ &\quad - (K_i(l) - K_i(l+1))^T R_i (K_i(l) - K_i(l+1)) \leq 0. \end{aligned}$$

Again applying Theorem 2.1, we conclude from the above equalities that

$$X_i(0) \geq X_i(1) \geq \cdots \geq X_i(l) \geq \tilde{X}_i \quad \text{for every } l \geq 0.$$

The sequences defined above are nonincreasing and bounded below. Thus, there exist N matrices X_i^+, $i = 1, \cdots, N$ such that $X_i(l)$ converges to X_i^+, $i = 1, \cdots, N$

$$X_i^+ = \lim_{l \to +\infty} X_i(l); \qquad X_i^+ \geq \tilde{X}_i.$$

By passing to the limit in the equalities

$$\begin{aligned} A_i(l)^T X_i(l) + X_i(l) A_i(l) + \sum_{j=1}^N \pi_{ij} X_j(l) \\ = -K_i(l)^T R_i K_i(l) - Q_i \end{aligned}$$

it follows that $\mathcal{R}_i(X_1^+, \cdots, X_N^+) = 0$, $i = 1, \cdots, N$. □

Case When $Q_i \geq 0$, $i = 1, \cdots, N$: Let $\tilde{X}_1, \cdots, \tilde{X}_N$ be any symmetric matrix satisfying $\mathcal{R}_i(\tilde{X}_1, \cdots, \tilde{X}_N) \geq 0$. Define for $\epsilon > 0$

$$\mathcal{R}_i^\epsilon(X_1, \cdots, X_N) = \mathcal{R}_i(X_1, \cdots, X_N) + \epsilon I.$$

Noticing that $Q_i + \epsilon I > 0$ we can apply the previous result to obtain matrices $X_i^+(\epsilon)$, the solution of $\mathcal{R}_i^\epsilon(X_1, \cdots, X_N) = 0$, and $X_i^+(\epsilon) \geq \tilde{X}_i; i = 1, \ldots, N$.

Now, the solutions $X_i^+(\epsilon)$ are nonincreasing with ϵ. That is, if $\epsilon_1 > \epsilon_2$, then $\mathcal{R}_i^{\epsilon_1}(X_1^+(\epsilon_2), \cdots, X_N^+(\epsilon_2)) \geq 0$ which implies $X_i^+(\epsilon_1) \geq X_i^+(\epsilon_2) \geq \tilde{X}_i$. In the sequence $X_i^+(\epsilon)$, being nonincreasing and bounded below by $\tilde{X}_i$, there exists X_i^+ such that $X_i^+ = \lim_{\epsilon \to 0} X(\epsilon)_i$ and $X_i^+ \geq \tilde{X}_i$.

Passing to the limit in the equations $\mathcal{R}_i^\epsilon(X_1^+(\epsilon), \cdots, X_N^+(\epsilon)) = 0$, we obtain

$$\mathcal{R}_i(X_1^+, \cdots, X_N^+) = 0.$$

Finally, the matrices X_i^+ are positive semidefinite since $\mathcal{R}_i(0, \cdots, 0) = Q_i \geq 0$. $\square$

ACKNOWLEDGMENT

The authors would like to thank the Associate Editor and the anonymous reviewers for their helpful comments.

REFERENCES

[1] H. Abou-Kandil, G. Freiling, and G. Jank, "Solution and asymptotic behavior of coupled Riccati equations in jump linear systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 8, pp. 1631–1636, Aug. 1994.

[2] ——, "On the solution of discrete-time Markovian jump linear quadratic control problems," *Automatica*, vol. 31, no. 5, pp. 765–768, May 1995.

[3] C. E. de Souza and M. D. Fragoso, "$\mathbf{H}^\infty$ control for linear systems with Markovian jumping parameters," in *Proc. European Contr. Conf.*, Grenoble, France, July 1991, pp. 17–21; also in *Contr. Theor. Adv. Technol.*, pp. 457–466, 1993.

[4] L. El Ghaoui, "State-feedback control of systems with multiplicative noise via linear matrix inequalities," *Syst. Contr. Lett.*, vol. 24, no. 3, pp. 223–228, Feb. 1995.

[5] L. El Ghaoui and M. Ait-Rami, "Robust state-feedback stabilization of jump linear systems," *Int. J. Robust Nonlinear Contr.*, to appear.

[6] L. El Ghaoui, R. Nikoukhah, and F. Delebecque, "LMITOOL: A Front-End for LMI Optimization—User's Guide," available via anonymous ftp to ftp.ensta.fr, under /pub/elghaoui/lmitool, Feb. 1995.

[7] A. S. Willsky, H. J. Chizeck, and D. Castanon, "Discrete-time Markovian-jump linear quadratic optimal control," *Int. J. Contr.*, vol. 43, no. 1, pp. 213–231, 1986.

[8] Y. Ji and H. J. Chizeck, "Controllability, observability and discrete-time Markovian jump linear quadratic control," *Int. J. Contr.*, vol. 48, no. 2, pp. 481–498, 1988.

[9] ——, "Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control," *IEEE Trans. Automat. Contr.*, vol. 35, no. 7, pp. 777–788, July 1990.

[10] A. G. F. MacFarlane, "An eigenvector solution of the optimal linear regulator problem," *J. Electron. Contr.*, vol. 14, pp. 491–501, 1963.

[11] M. Mariton, "Jump linear quadratic control with random state discontinuities," *Automatica*, vol. 23, no. 2, pp. 237–240, 1987.

[12] ——, "On the influence of noise on jump linear systems," *IEEE Trans. Automat. Contr.*, vol. 32, no. 12, pp. 1094–1097, Dec. 1987.

[13] M. Mariton and P. Bertrand, "A homotopy algorithm for solving coupled Riccati equations," *Opt. Contr. Appl. Methods*, vol. 6, pp. 351–357, 1985.

[14] Yu. Nesterov and A. Nemirovsky, "Interior point polynomial methods in convex programming: Theory and applications," *SIAM*, 1994.

[15] J. E. Potter, "Matrix quadratic solutions," *J. SIAM Appl. Math.*, vol. 14, no. 3, pp. 496–501, May 1966.

[16] L. Vandenberghe and S. Boyd, "Software for semidefinite programming—User's guide," available via anonymous ftp to isl.stanford.edu under /pub/boyd/semidef_prog, Dec. 1994.

[17] ——, "Semidefinite programming," *SIAM Rev.*, vol. 38, no. 1, pp. 49–95, Mar. 1996.

[18] W. M. Wonham, "Random differential equations in control theory," in *Probabilistic Methods in Appl. Math.*, A. T. Bharucha-Reid, Ed., vol. 2. New York: Academic, 1970, pp. 131–212.

[19] W.-Y. Yan, J. B. Moore, and U. Helmke, "Recursive algorithms for solving a class of nonlinear matrix equations with applications to certain sensitivity optimization problems," *SIAM J. Contr. Optimization*, vol. 32, no. 6, pp. 1559–1576, Nov. 1994.

A Gradient Algorithm for Stereo Matching Without Correspondence

J. Zhou and B. K. Ghosh

Abstract—**In this paper we study the problem of feature point matching via a technique well known in geometry, leading to a new gradient algorithm in machine vision. The procedure does not require one to solve explicit correspondences between feature points but relies on a specific form of "spatial averaging." In this sense the estimation procedure is robust. In this paper we describe the algorithm and report simulation results.**

I. INTRODUCTION

In this paper we propose a new gradient algorithm for stereo correspondence of a set of points on a plane, the points being observed by a pair of CCD cameras. Our goal is to compute the position of the plane on which the point lies without explicitly computing the exact three-dimensional (3-D) coordinates of the points individually. Thus if an unknown surface is observed via a pair of cameras, the procedure would generate a local description of the "shape" of the surface without explicitly identifying the locations of the features in space. Of course, once the surface has been identified, points in each of the two images can be corresponded easily. The proposed algorithm relies heavily on a new gradient algorithm on a Lie group described in the two papers by Brockett [1], [2]. We would also like to recommend a recent monograph written by Helmke and Moore [6] on gradient flows.

To motivate the proposed algorithm of this paper, we shall recall first of all the "matching problem" from [1]. Let us consider an unordered collection of a set of points in $\mathbb{R}^3$ denoted by $\{x_1, x_2, \cdots, x_s\}$, where $x_i \in \mathbb{R}^3, i = 1, \cdots, s$. Furthermore, we assume that the points undergo a rigid transformation via a rotation matrix. Specifically, we define the map

$$\begin{aligned} \mathcal{R}&: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \\ x_i &\mapsto Rx_i = y_i \end{aligned} \tag{1}$$

where R is a rotation matrix, i.e., R is an element of $SO(3)$, where

$$SO(3) \triangleq \{R \in \mathbb{R}^{3\times 3}; R^T R = I, \det R = 1\}. \tag{2}$$

The group of matrices $SO(3)$ is known as the special orthogonal group. We assume that under the above map (1) the set of points

$$S_1 \triangleq \{x_1, x_2, \cdots, x_s\} \tag{3}$$

is mapped to the set of points

$$S_2 \triangleq \{y_1, y_2, \cdots, y_s\}. \tag{4}$$

Thus each $x_i, i \in \{1, \cdots, s\}$ is mapped to some $x_j, j \in \{1, \cdots, s\}$ under the map (1). The matching problem analyzed by Brockett [1] can be described as follows.

Manuscript received November 21, 1994. This work was supported in part by DOE under Grant DE-FG02-90ER14140.

J. Zhou is with Broadway and Seymour, Inc., Charlotte, NC 29202-5050 USA.

B. K. Ghosh is with the Department of Systems Science and Mathematics, Washington University, St. Louis, MO 63130-4899 USA (e-mail: ghosh@zach.wustl.edu).

Publisher Item Identifier S 0018-9286(96)07510-1.