Optimization and (Under/over)fitting

EECS 442 – David Fouhey

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http://web.eecs.umich.edu/~fouhey/teaching/EECS442_F19/

Regularized Least Squares

Add **regularization** to objective that prefers some solutions:

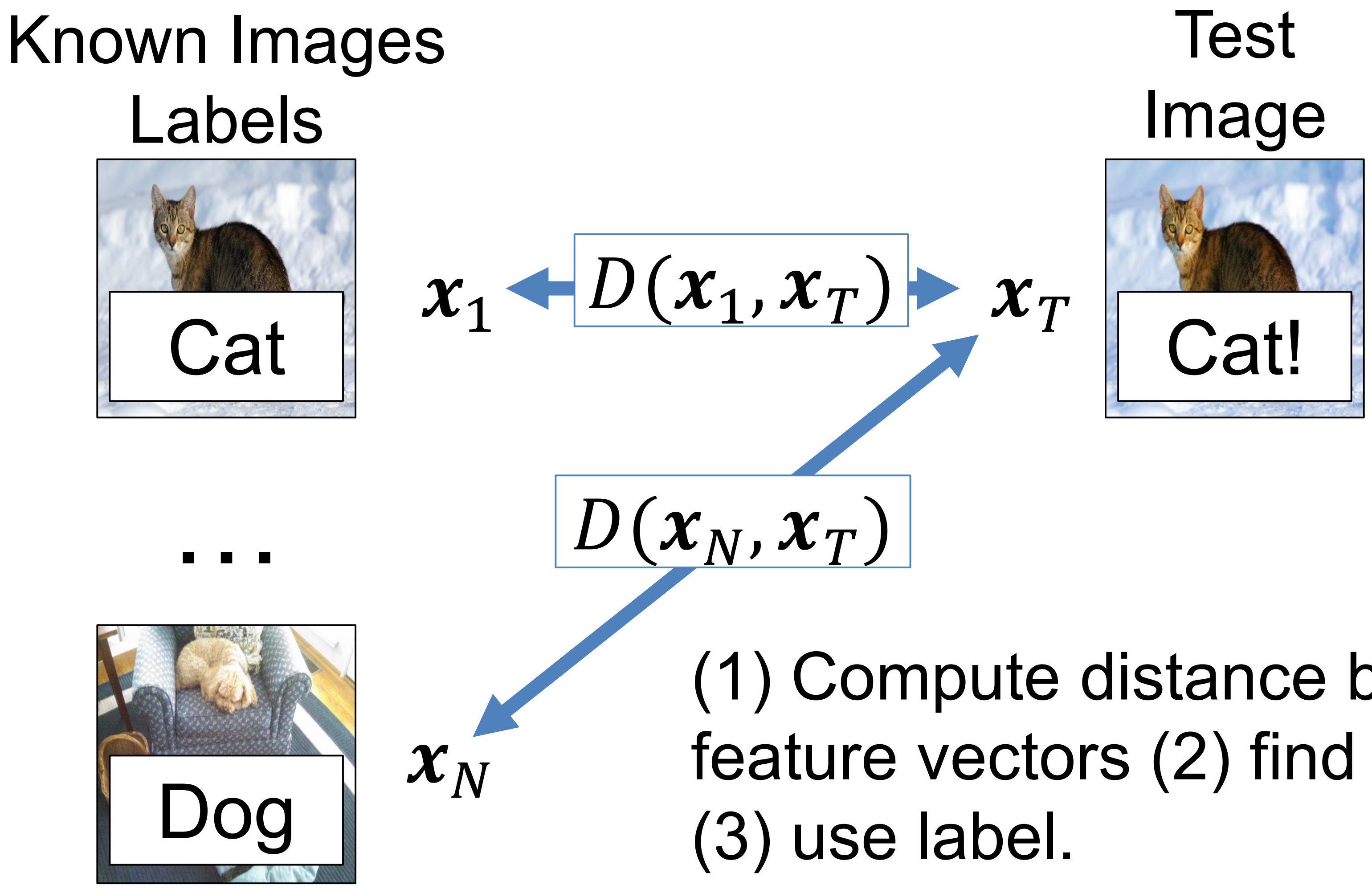
Before: $$\arg\min_{\boldsymbol{w}} \|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}\|_2^2 \longrightarrow \text{Loss}$$

After: $$\arg\min_{\boldsymbol{w}} \underbrace{\|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}\|_2^2}_{\text{Loss}} + \underbrace{\lambda}_{\text{Trade-off}} \underbrace{\|\boldsymbol{w}\|_2^2}_{\text{Regularization}}$$

Want model "smaller": pay a penalty for **w** with big norm

Intuitive Objective: accurate model (low loss) but not too complex (low regularization). λ controls how much of each.

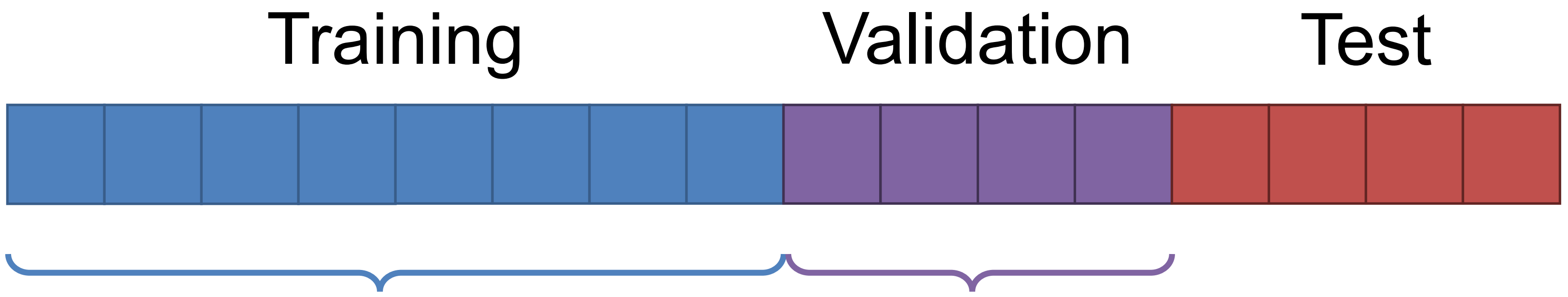
Nearest Neighbors



(1) Compute distance between feature vectors (2) find nearest (3) use label.

Picking Parameters

What distance? What value for k / λ?



Use these data points for lookup

Evaluate on these points for different k, λ, distances

Linear Models

Example Setup: 3 classes

Model – one weight per class: $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2$

Want:

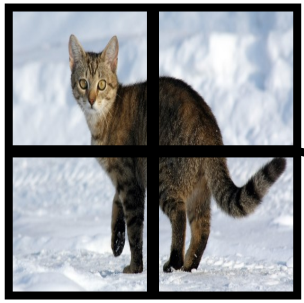
$\boldsymbol{w}_0^T \boldsymbol{x}$ big if cat

$\boldsymbol{w}_1^T \boldsymbol{x}$ big if dog

$\boldsymbol{w}_2^T \boldsymbol{x}$ big if hippo

Stack together: $\boldsymbol{W}_{\mathbf{3xF}}$ where **x** is in R^F

Linear Models



$$\begin{array}{r|c|c|c|c|c|} \hline \text{Cat weight vector} & 0.2 & -0.5 & 0.1 & 2.0 & 1.1 \\ \hline \text{Dog weight vector} & 1.5 & 1.3 & 2.1 & 0.0 & 3.2 \\ \hline \text{Hippo weight vector} & 0.0 & 0.3 & 0.2 & -0.3 & -1.2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 56 \\ \hline 231 \\ \hline 24 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|l} \hline -96.8 & \text{Cat score} \\ \hline 437.9 & \text{Dog score} \\ \hline 61.95 & \text{Hippo score} \\ \hline \end{array}$$

$\boldsymbol{W}$ $\quad$ $\boldsymbol{x_i}$ $\quad$ $\boldsymbol{Wx_i}$

Weight matrix a collection of scoring functions, one per class

Prediction is vector where jth component is "score" for jth class.

Diagram by: Karpathy, Fei-Fei

Objective 1: Multiclass SVM*

Inference (x,y): $\arg\max_{k} \ (\boldsymbol{W}\boldsymbol{x})_k$

(Take the class whose weight vector gives the highest score)

Training ($\mathbf{x}_i$,y_i):

$$\arg\min_{\boldsymbol{W}} \lambda \|\boldsymbol{W}\|_2^2 + \sum_{i=1}^{n} \sum_{j \neq y_i} \underbrace{\max(0, (\boldsymbol{W}\boldsymbol{x}_i)_j - (\boldsymbol{W}\boldsymbol{x}_i)_{y_i})}$$

Regularization

Over all data points

For every class j that's NOT the correct one (y_i)

Pay no penalty if prediction for class y_i is bigger than j. Otherwise, pay proportional to the score of the wrong class.

Objective 1: Multiclass SVM

Inference (x,y): $\arg\max_{k} \ (\boldsymbol{W}\boldsymbol{x})_k$

(Take the class whose weight vector gives the highest score)

Training ($\mathbf{x}_i$,y_i):

$$\arg\min_{\boldsymbol{W}} \lambda \|\boldsymbol{W}\|_2^2 + \sum_{i=1}^{n} \sum_{j \neq y_i} \underbrace{\max(0, (\boldsymbol{W}\boldsymbol{x}_i)_j - (\boldsymbol{W}\boldsymbol{x}_i)_{y_i} + m)}$$

Regularization

Over all data points

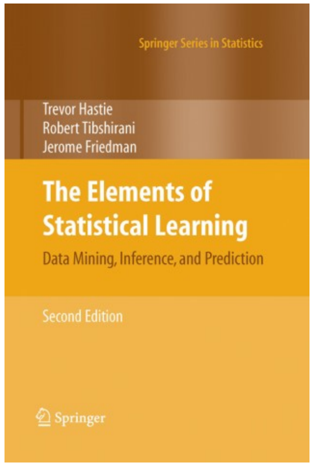
For every class j that's NOT the correct one (y_i)

Pay no penalty if prediction for class y_i is bigger than j by m ("margin"). Otherwise, pay proportional to the score of the wrong class.

Objective 1:

Called: Support Vector Machine

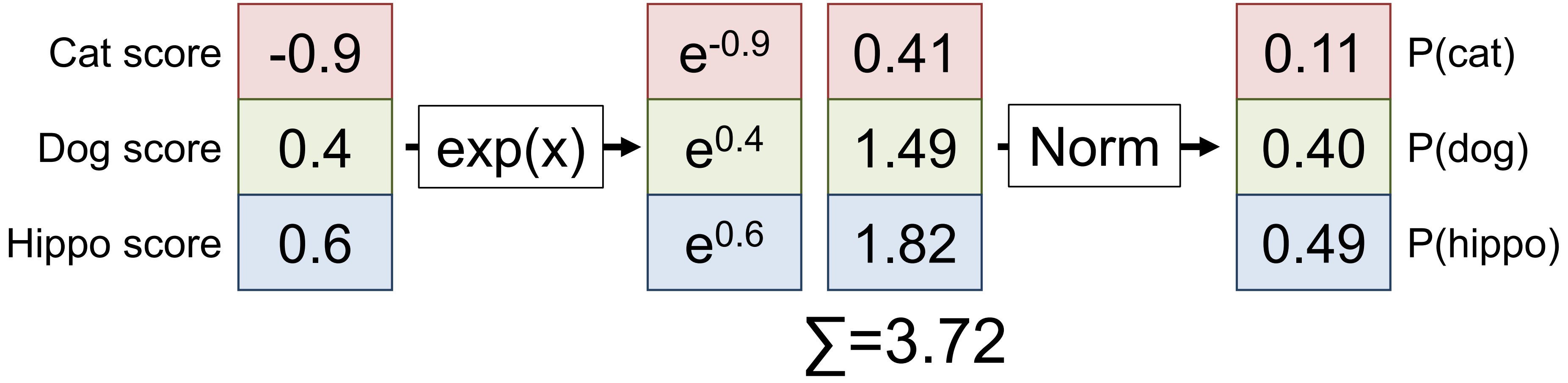
Lots of great theory as to why this is a sensible thing to do. See


Springer Series in Statistics
Trevor Hastie
Robert Tibshirani
Jerome Friedman
The Elements of Statistical Learning
Data Mining, Inference, and Prediction
Second Edition
Springer

Useful book (Free too!):
The Elements of Statistical Learning
Hastie, Tibshirani, Friedman
https://web.stanford.edu/~hastie/ElemStatLearn/

Objective 2: Making Probabilities

Converting Scores to “Probability Distribution”



Generally P(class j): $$\frac{\exp((Wx)_j)}{\sum_k \exp((Wx)_k)}$$

Called softmax function

Inspired by: Karpathy, Fei-Fei

Objective 2: Softmax

Inference (x): $\arg\max_{\mathrm{k}} \; (\boldsymbol{W}\boldsymbol{x})_k$

(Take the class whose weight vector gives the highest score)

P(class j): $$\frac{\exp((Wx)_j)}{\sum_k \exp((Wx)_k)}$$

Why can we skip the exp/sum exp thing to make a decision?

Objective 2: Softmax

Inference (x): $\arg\max_{k} \ (\boldsymbol{Wx})_k$

(Take the class whose weight vector gives the highest score)

Training ($\mathbf{x}_i$,y_i):

$$\underbrace{\arg\min_{\boldsymbol{W}} \lambda\|\boldsymbol{W}\|_2^2}_{\text{Regularization}} + \underbrace{\sum_{i=1}^{n}}_{\text{Over all data points}} \underbrace{-\log\left(\frac{\exp((Wx)_{y_i})}{\sum_k \exp((Wx)_k))}\right)}_{\text{Pay penalty for not making correct class likely. “Negative log-likelihood”}}$$

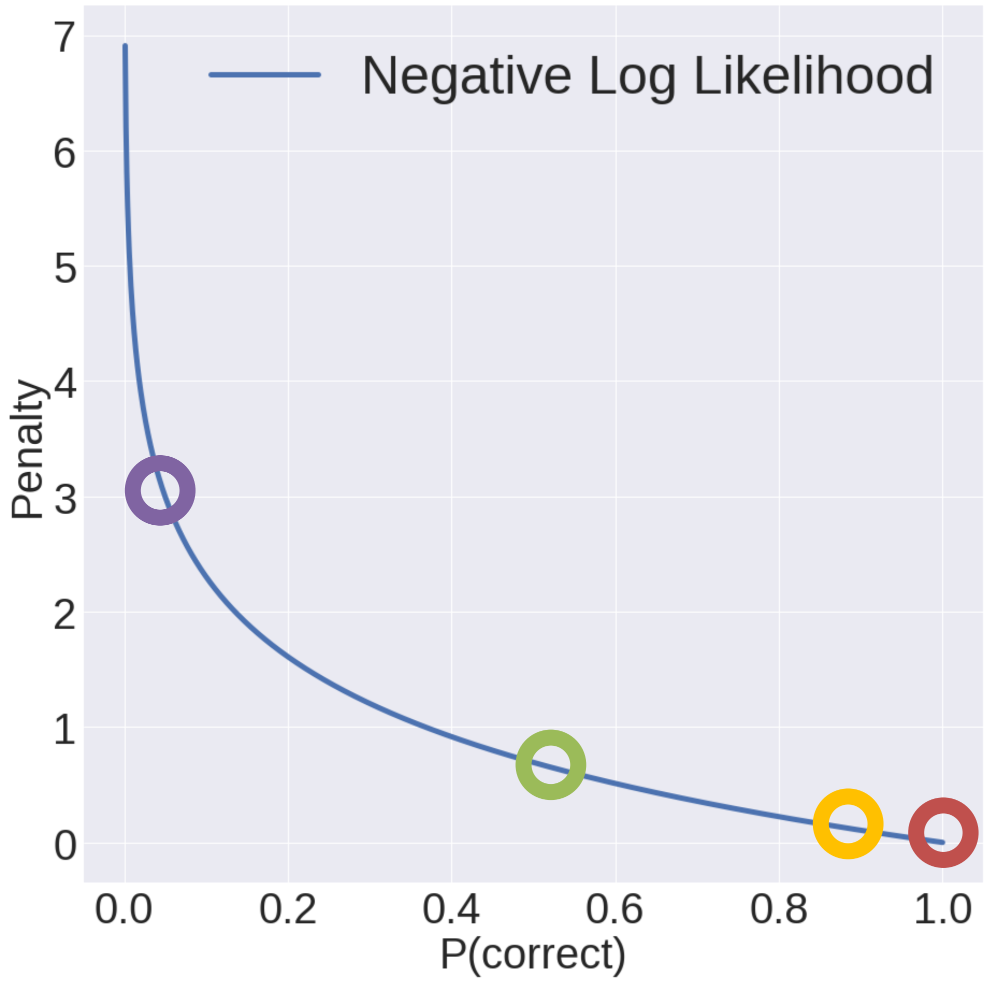
P(correct class)

Regularization

Over all data points

Pay penalty for not making correct class likely.
“Negative log-likelihood”

Objective 2: Softmax



P(correct) = 0.05: 3.0 penalty

P(correct) = 0.5: 0.11 penalty

P(correct) = 0.9: 0.11 penalty

P(correct) = 1: No penalty!

How Do We Optimize Things?

Goal: find the **w** minimizing some loss function L.

$$\arg\min_{\boldsymbol{w} \in R^N} L(\boldsymbol{w})$$

Works for lots of different Ls:

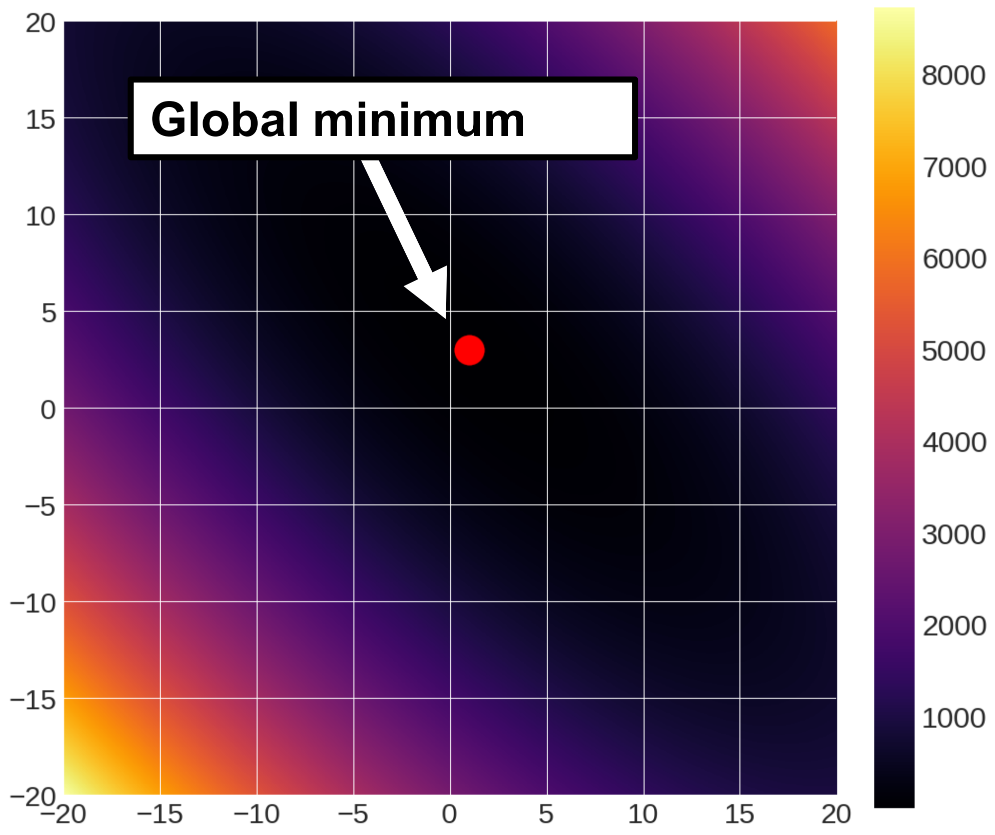
$$L(\boldsymbol{W}) = \boldsymbol{\lambda} \|\boldsymbol{W}\|_{\boldsymbol{2}}^{\boldsymbol{2}} + \sum_{i=1}^{n} -\log\left(\frac{\exp((Wx)_{y_i})}{\sum_k \exp((Wx)_k))}\right)$$

$$L(\boldsymbol{w}) = \lambda \|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n} \left(y_i - \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{x}_{\boldsymbol{i}}\right)^2$$

$$L(\boldsymbol{w}) = C \|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n} \max(0, 1 - y_i \boldsymbol{w}^T \boldsymbol{x}_{\boldsymbol{i}})$$

Sample Function to Optimize

$$f(x,y) = (x+2y-7)^2 + (2x+y-5)^2$$

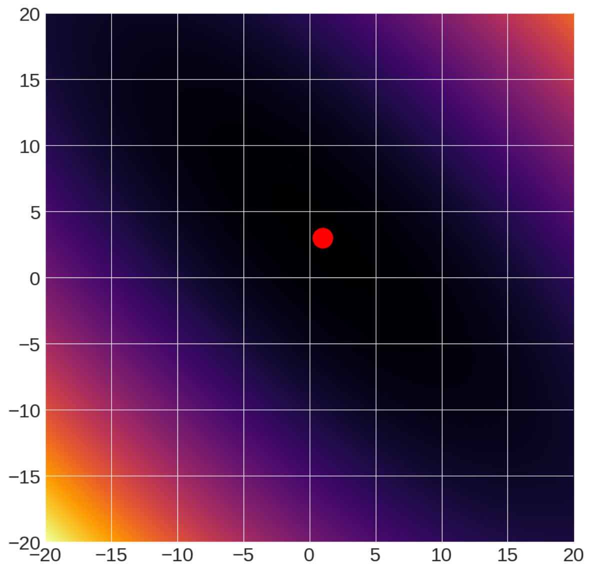


Sample Function to Optimize

- I’ll switch back and forth between this 2D function (called the *Booth Function*) and other more-learning-focused functions
- Beauty of optimization is that it’s all the same in principle
- But don’t draw too many conclusions: 2D space has qualitative differences from 1000D space

See intro of: Dauphin et al. *Identifying and attacking the saddle point problem in high-dimensional non-convex optimization* NIPS 2014
https://ganguli-gang.stanford.edu/pdf/14.SaddlePoint.NIPS.pdf

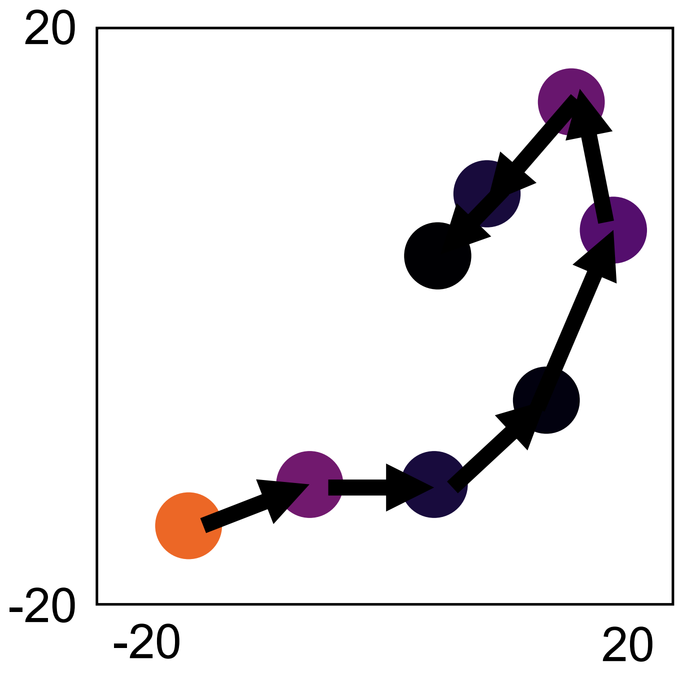
A Caveat

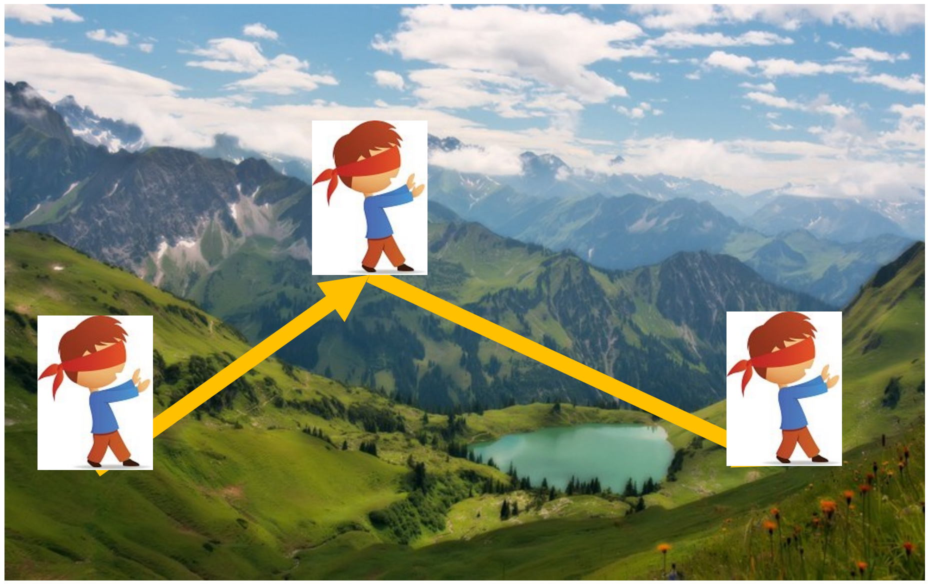


- Each point in the picture is a function evaluation
- Here it takes microseconds – so we can easily see the answer
- Functions we want to optimize may take hours to evaluate

A Caveat

Model in your head: moving around a landscape with a teleportation device



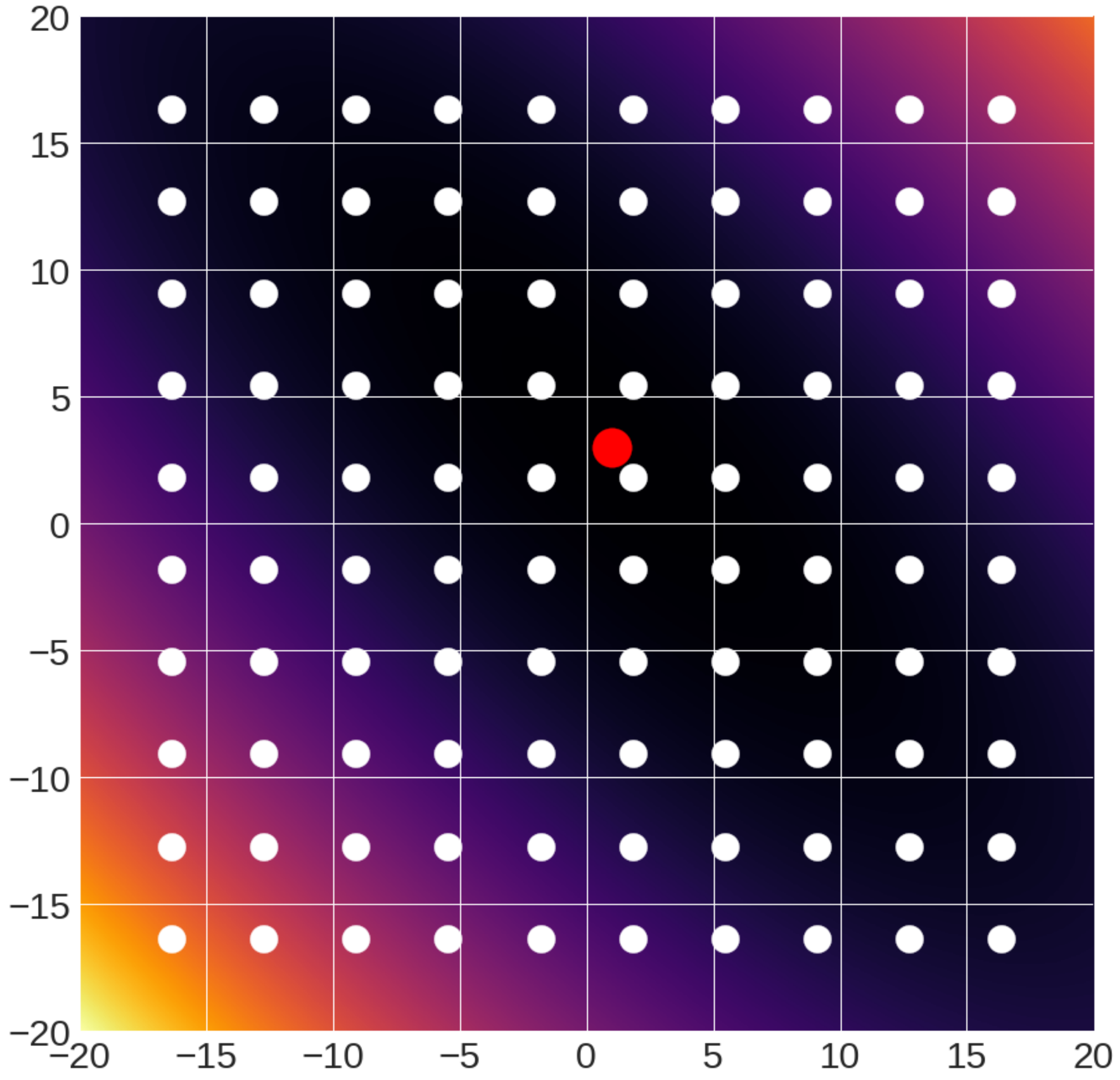


Landscape diagram: Karpathy and Fei-Fei

Option #1A – Grid Search

```
#systematically try things
best, bestScore = None, Inf
for dim1Value in dim1Values:
    ….
    for dimNValue in dimNValues:
        w = [dim1Value, …, dimNValue]
        if L(w) < bestScore:
            best, bestScore = w, L(w)
return best
```

Option #1A – Grid Search



Option #1A – Grid Search

Pros:

1. Super simple
2. Only requires being able to evaluate model

Cons:

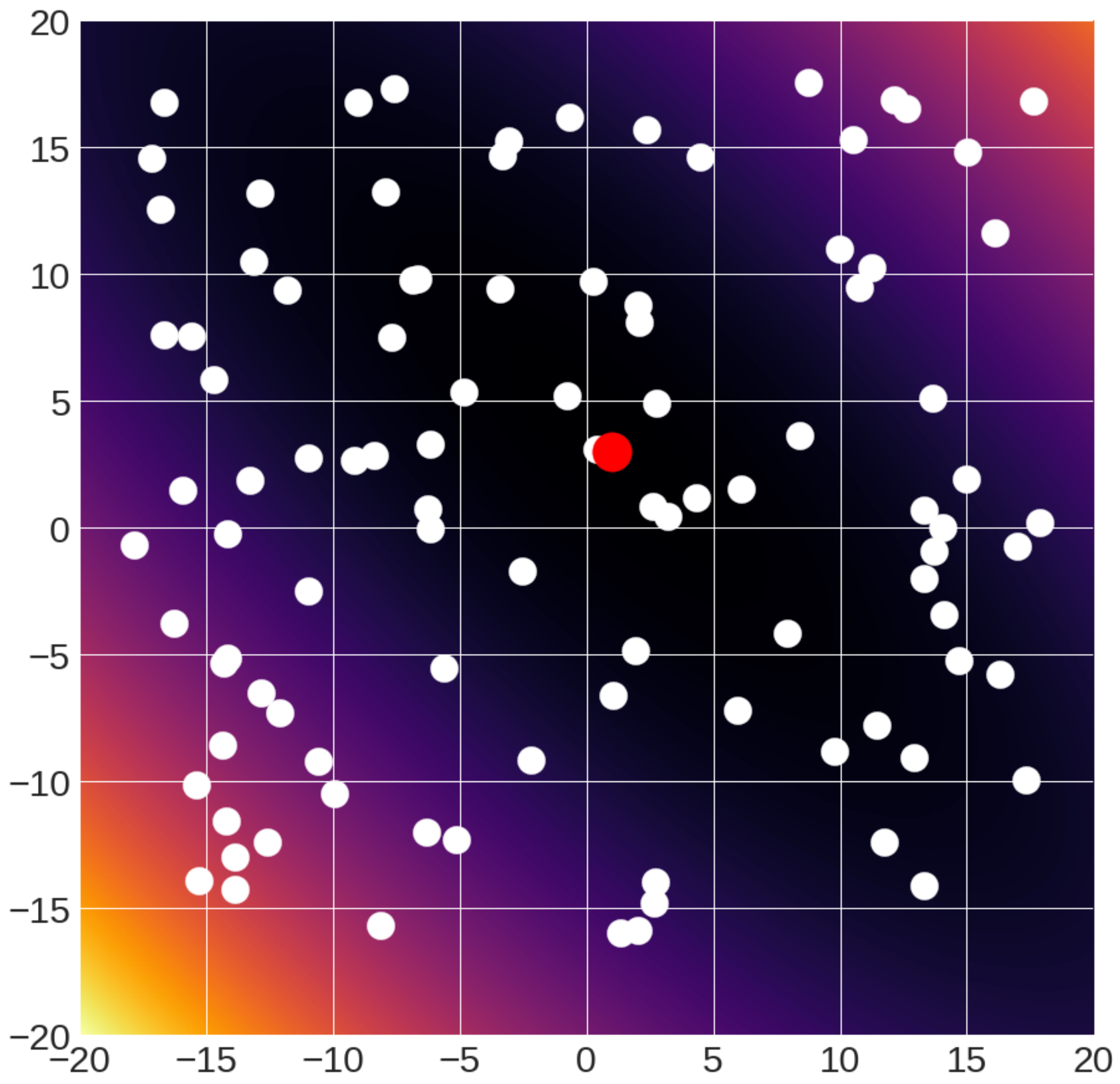
1. Scales horribly to high dimensional spaces

Complexity: $\text{samplesPerDim}^{\text{numberOfDims}}$

Option #1B – Random Search

```
#Do random stuff RANSAC Style
best, bestScore = None, Inf
for iter in range(numIters):
        w = random(N,1) #sample
        score = $L(\boldsymbol{w})$ #evaluate
        if score < bestScore:
                best, bestScore = w, score
return best
```

Option #1B – Random Search



Option #1B – Random Search

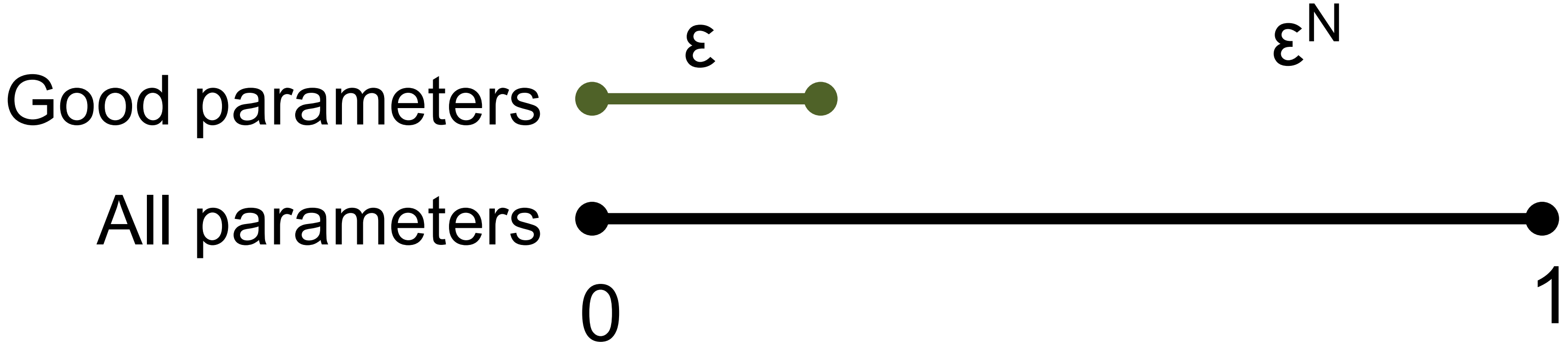
Pros:

1. Super simple
2. Only requires being able to sample model and evaluate it

Cons:

1. Slow –throwing darts at high dimensional dart board
2. Might miss something

P(all correct) = ε^N



When Do You Use Options 1A/1B?

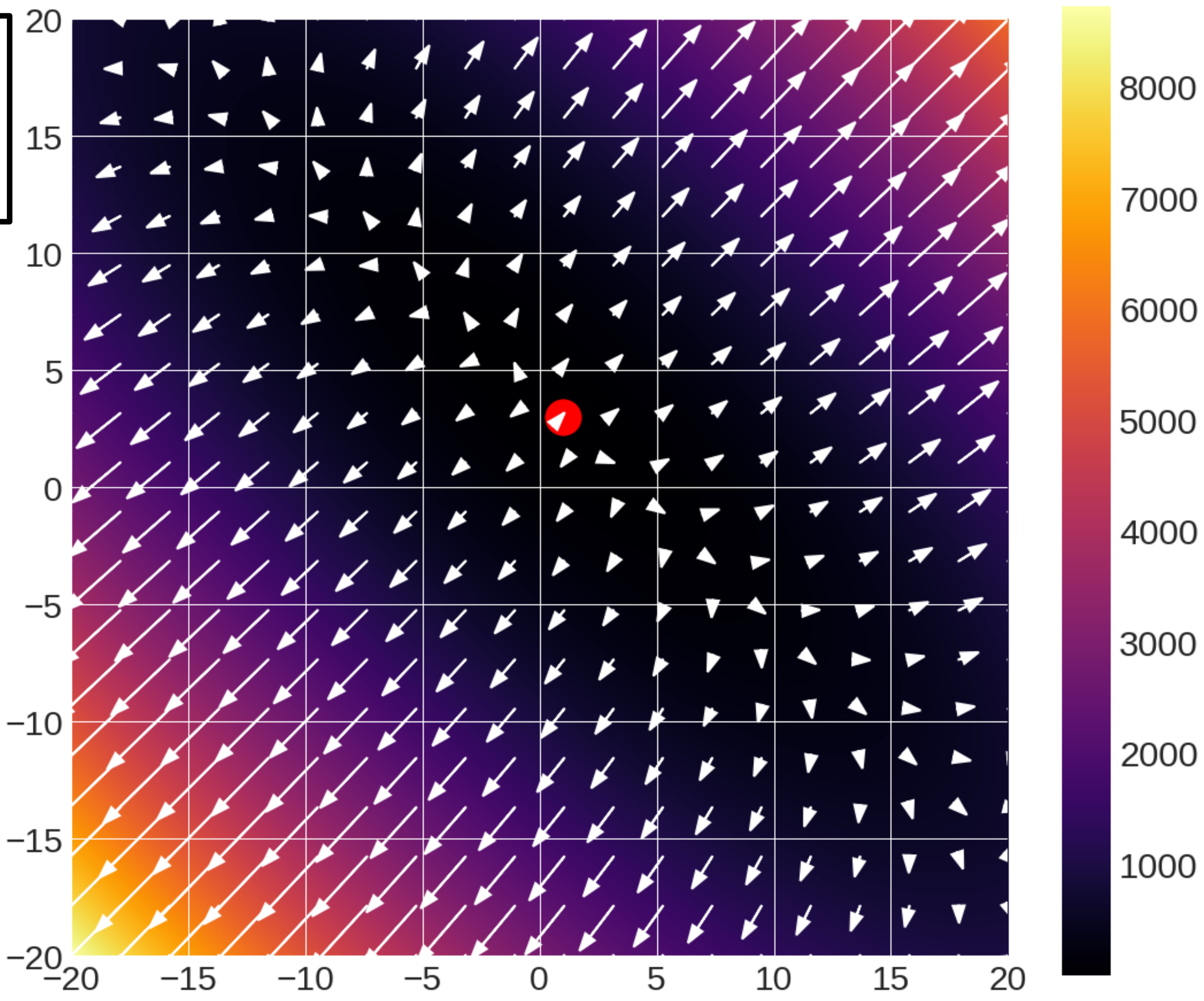
Use these when

- Number of dimensions small, space bounded
- Objective is impossible to analyze (e.g., test accuracy if we use this distance function)

Random search is arguably more effective; grid search makes it easy to systematically test something (people love certainty)

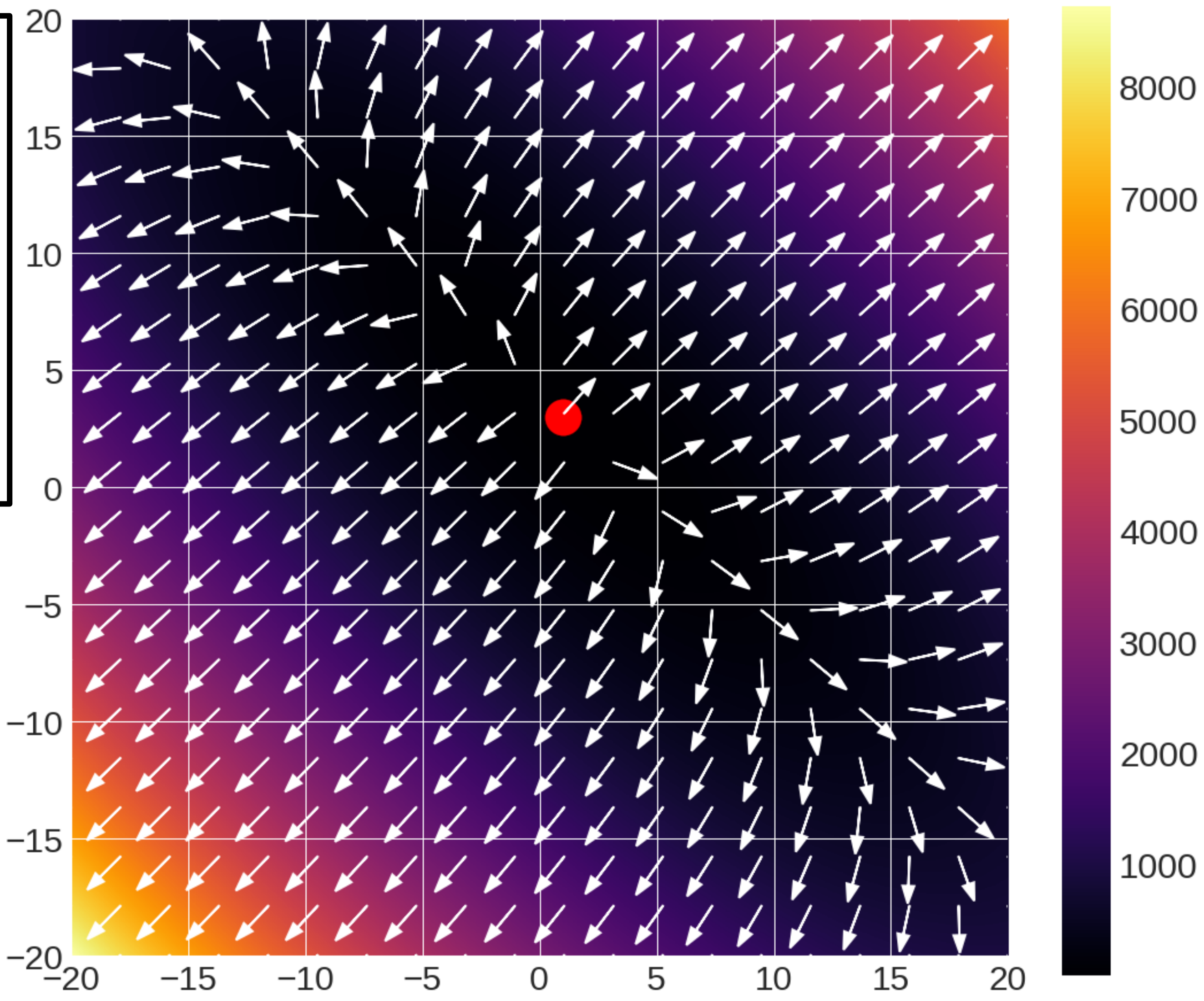
Option 2 – Use The Gradient

Arrows:
gradient



Option 2 – Use The Gradient

Arrows:
gradient direction
(scaled to unit length)



Option 2 – Use The Gradient

Want: $\arg\min_{\boldsymbol{w}} L(\boldsymbol{w})$

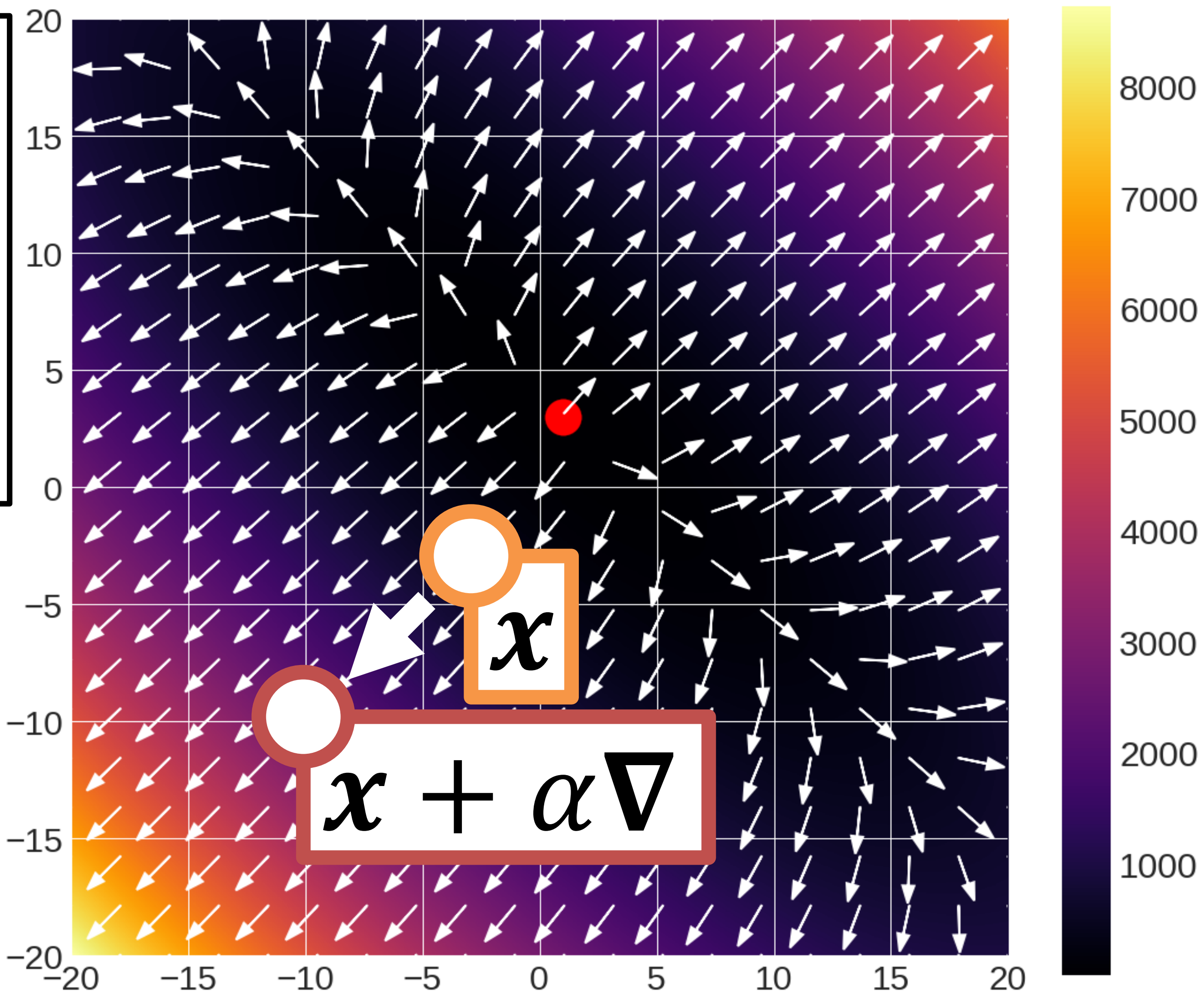
What's the geometric interpretation of: $\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \begin{bmatrix} \partial L / \partial \boldsymbol{x}_1 \\ \vdots \\ \partial L / \partial \boldsymbol{x}_N \end{bmatrix}$

Which is bigger (for small α)?

$$L(\boldsymbol{w}) \quad \begin{matrix} \leq ? \\ \\ >? \end{matrix} \quad L(\boldsymbol{w} + \alpha \nabla_{\boldsymbol{w}} L(\boldsymbol{w}))$$

Option 2 – Use The Gradient

Arrows: gradient direction (scaled to unit length)



Option 2 – Use The Gradient

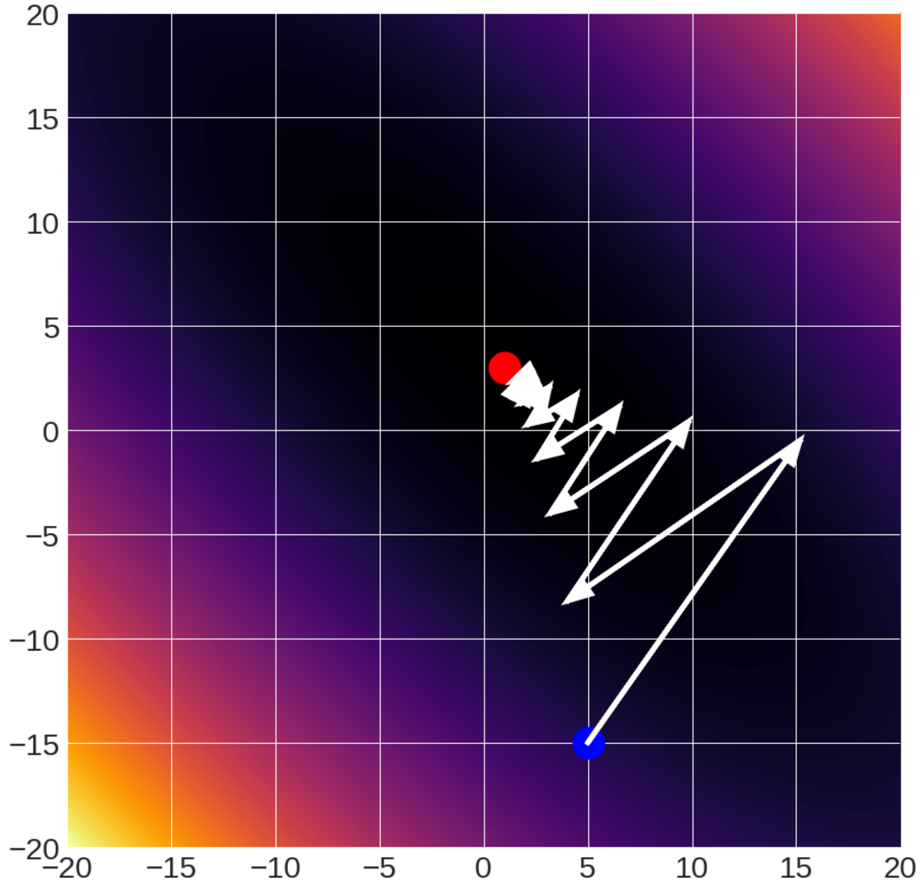
Method: at each step, move in direction of negative gradient

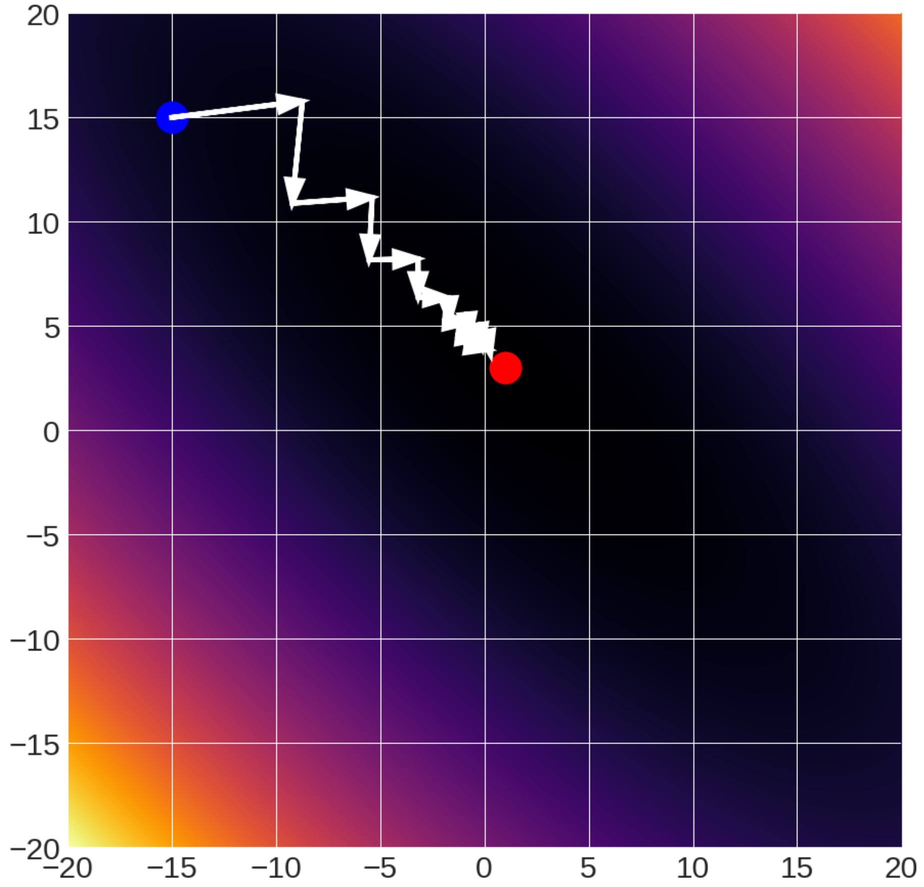
```
**w0** = *initialize*() #initialize
for iter in range(numIters):
    **g** = $\nabla_{\boldsymbol{w}} L(\boldsymbol{w})$ #eval gradient
    **w** = **w** + -*stepsize*(iter)***g** #update w
return **w**
```

Gradient Descent

Given starting point (blue)

$w_{i+1} = w_i + -9.8\text{x}10^{-2}$ x gradient





Computing the Gradient

How Do You Compute The Gradient? Numerical Method:

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \begin{bmatrix} \boxed{\dfrac{\partial L(w)}{\partial x_1}} \\ \vdots \\ \dfrac{\partial L(w)}{\partial x_n} \end{bmatrix}$$

How do you compute this?

$$\frac{\partial f(x)}{\partial x} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

In practice, use:

$$\frac{f(x+\epsilon) - f(x-\epsilon)}{2\epsilon}$$

Computing the Gradient

How Do You Compute The Gradient?
Numerical Method:

$$\nabla_{w} L(\boldsymbol{w}) = \begin{bmatrix} \boxed{\dfrac{\partial L(w)}{\partial x_1}} \\ \vdots \\ \dfrac{\partial L(w)}{\partial x_n} \end{bmatrix}$$

Use: $\dfrac{f(x+\epsilon) - f(x-\epsilon)}{2\epsilon}$

How many function evaluations per dimension?

Computing the Gradient

How Do You Compute The Gradient?

Analytical Method:

$$\nabla_{w} L(\boldsymbol{w}) = \begin{bmatrix} \frac{\partial L(w)}{\partial x_1} \\ \vdots \\ \frac{\partial L(w)}{\partial x_n} \end{bmatrix}$$

Use calculus!

Computing the Gradient

$$L(\boldsymbol{w}) = \lambda \|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n} \left(y_i - \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{x_i}\right)^2$$

$$\Big\downarrow \quad \frac{\partial}{\partial \boldsymbol{w}} \quad \Big\downarrow$$

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \; 2\lambda \boldsymbol{w} + \sum_{i=1}^{n} -(2(y_i - \boldsymbol{w}^T \boldsymbol{x_i})\boldsymbol{x}_i)$$

Note: if you look at other derivations, things are written either (y-w^{T}x) or (w^{T}x – y); the gradients will differ by a minus.

Interpreting Gradients (1 Sample)

Recall:

w = **w** + -$\nabla_{\boldsymbol{w}} L(\boldsymbol{w})$ #update w

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = 2\lambda \boldsymbol{w} + -(2(y - \boldsymbol{w}^T \boldsymbol{x})\boldsymbol{x})$$

$$-\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \overbrace{-2\lambda \boldsymbol{w}}^{\text{Push } \mathbf{w} \text{ towards 0}} + \underbrace{\overbrace{(2(y - \boldsymbol{w}^T \boldsymbol{x})}^{\alpha}\boldsymbol{x})}$$

If $y > w^T x$ (too *low*): then $w = w + \alpha x$ for some α

Before: $w^T x$

After: $(w + \alpha x)^T x = w^T x + \alpha x^T x$

Quick annoying detail: subgradients

What is the derivative of |x|?

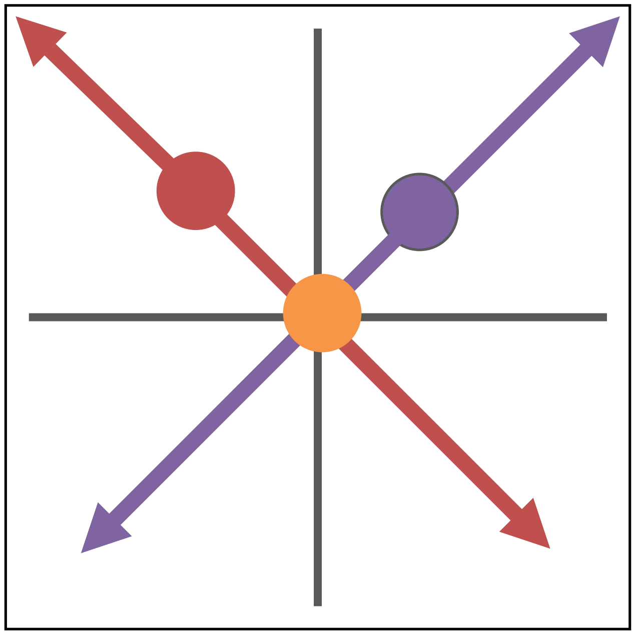
<u>Derivatives/Gradients</u>

Defined everywhere but 0

$$\frac{\partial}{\partial x} f(x) = \text{sign}(x) \quad x \neq 0$$

$$\text{undefined} \quad x = 0$$

|x|



Oh no! A discontinuity!

Quick annoying detail: subgradients

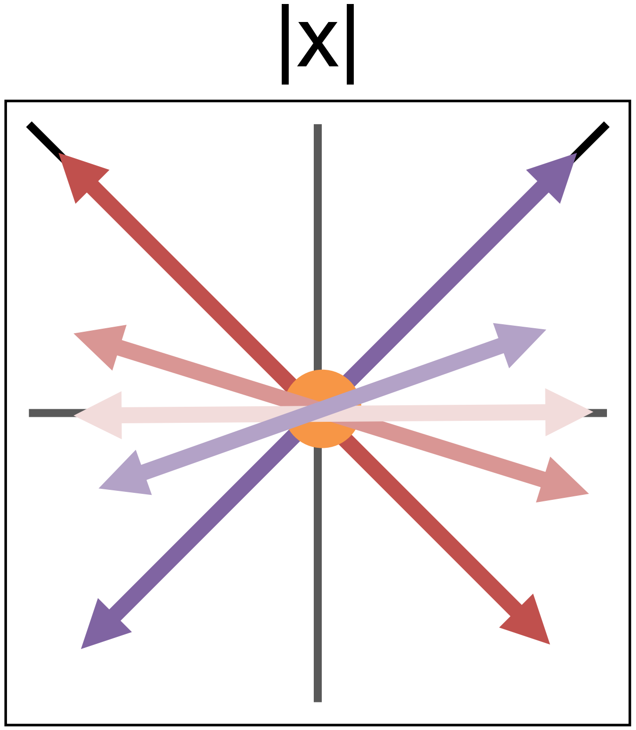
Subgradient: any underestimate of function

Subderivatives/subgradients

Defined everywhere

$$\frac{\partial}{\partial x} f(x) = \text{sign}(x) \quad x \neq 0$$

$$\frac{\partial}{\partial x} f(x) \in [-1,1] \quad x = 0$$



In practice: at discontinuity, pick value on either side.

Computing The Gradient

- Numerical: foolproof but slow
- Analytical: can mess things up ☺
- In practice: do analytical, but check with numerical (called a gradient check)

Slide: Karpathy and Fei-Fei

Implementing Gradient Descent

Loss is a function that we can evaluate over data

All Data

$$-\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = -2\lambda \boldsymbol{w} + \sum_{i=1}^{n} (2(y_i - \boldsymbol{w}^T \boldsymbol{x_i})\boldsymbol{x}_i)$$

Subset B

$$-\nabla_{\boldsymbol{w}} L_B(\boldsymbol{w}) = -2\lambda \boldsymbol{w} + \sum_{i \in B} (2(y_i - \boldsymbol{w}^T \boldsymbol{x_i})\boldsymbol{x}_i)$$

Implementing Gradient Descent

Option 1: Vanilla Gradient Descent
Compute gradient of L over all data points

```
for iter in range(numIters):
    g = gradient(data,L)
    w = w + -stepsize(iter)*g #update w
```

Implementing Gradient Descent

Option 2: *Stochastic* Gradient Descent
Compute gradient of L over 1 random sample

```
for iter in range(numIters):
    index = randint(0,#data)
    g = gradient(data[index],L)
    w = w + -stepsize(iter)*g #update w
```

Implementing Gradient Descent

Option 3: *Minibatch* Gradient Descent
Compute gradient of L over subset of B samples

```
for iter in range(numIters):
    subset = choose_samples(#data,B)
    g = gradient(data[subset],L)
    w = w + -stepsize(iter)*g #update w
```

Typical batch sizes: ~100

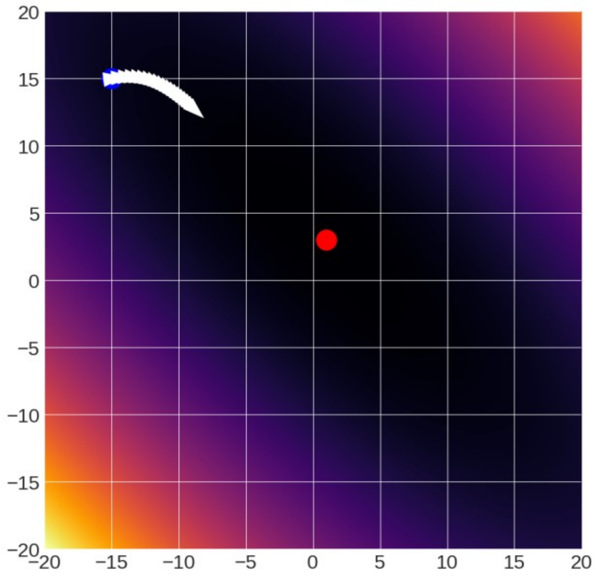
Gradient Descent Details

Step size (also called **learning rate / lr**)

critical parameter

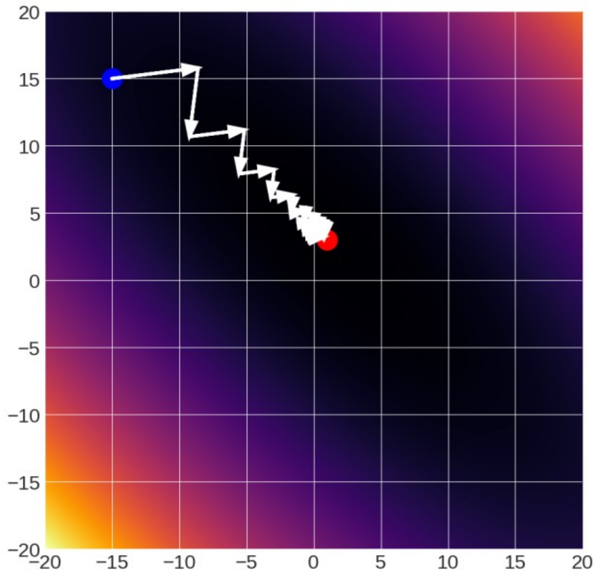
$1\text{x}10^{-2}$

falls short



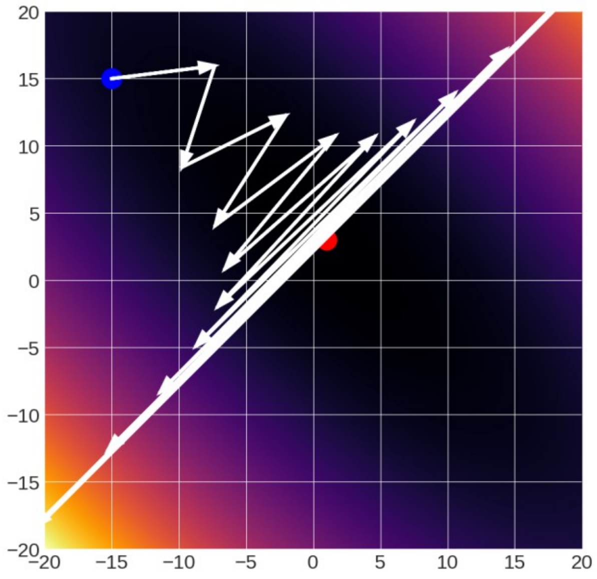
$10\text{x}10^{-2}$

converges



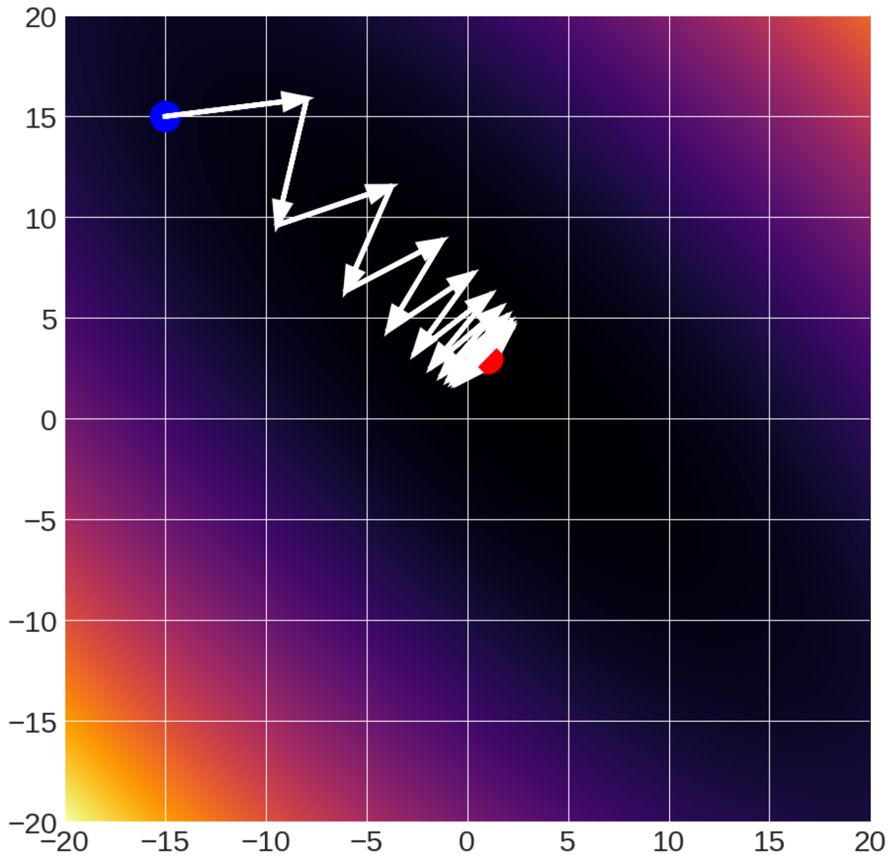
$12\text{x}10^{-2}$

diverges



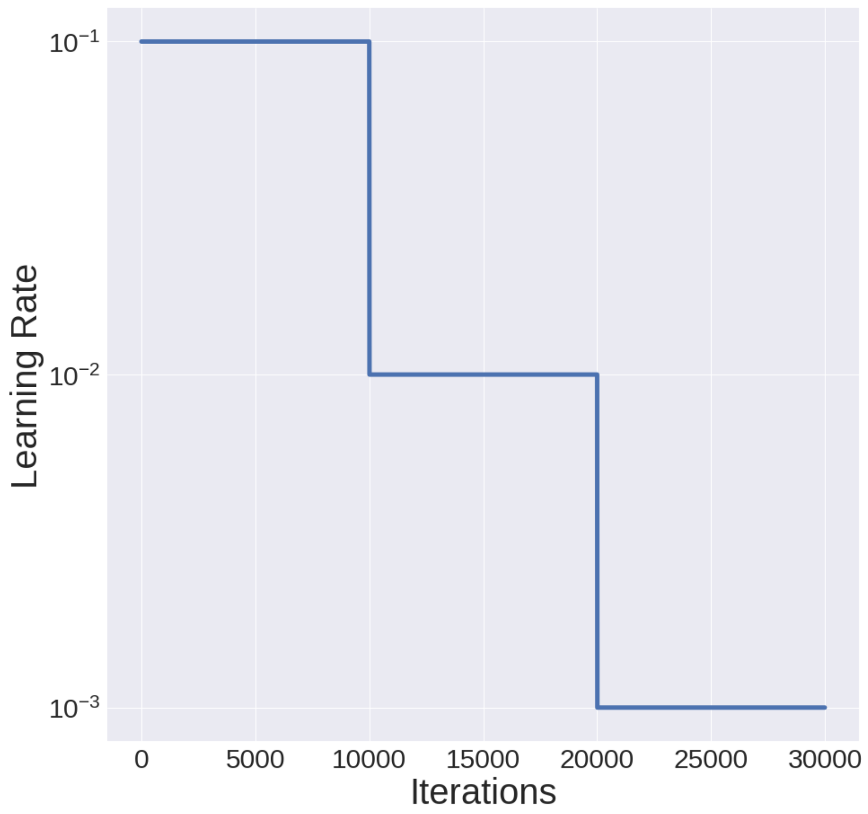
Gradient Descent Details

11×10^{-2} :oscillates
(Raw gradients)



Gradient Descent Details

One solution: start with initial rate lr, multiply by f every N interations



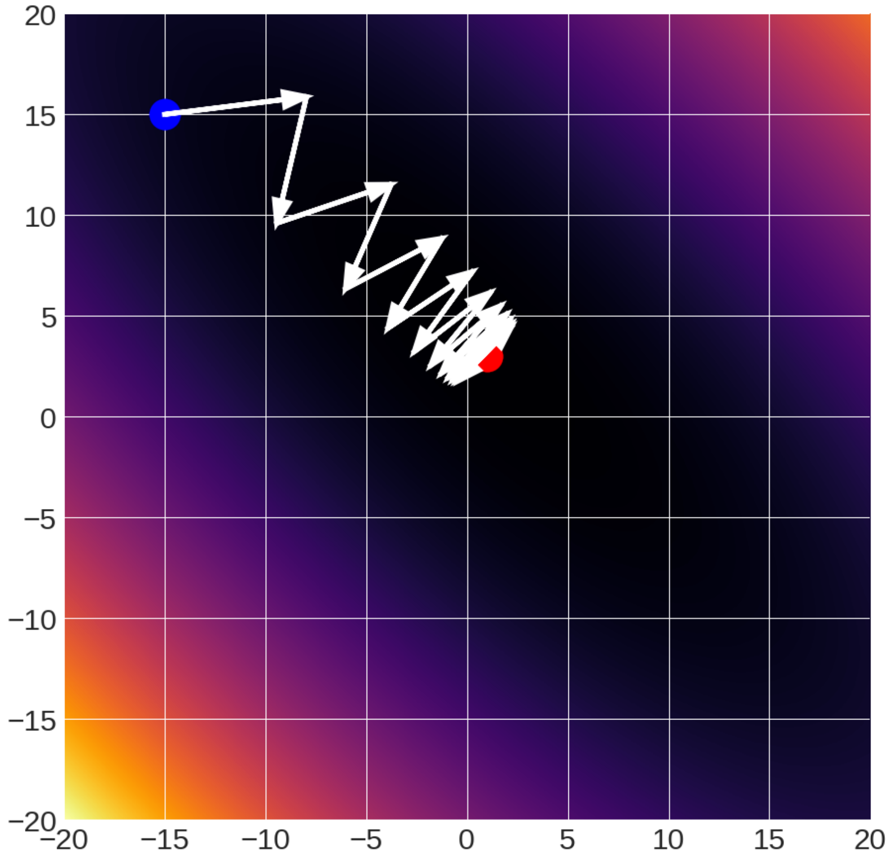
init_lr = 10^{-1}

f = 0.1

N = 10K

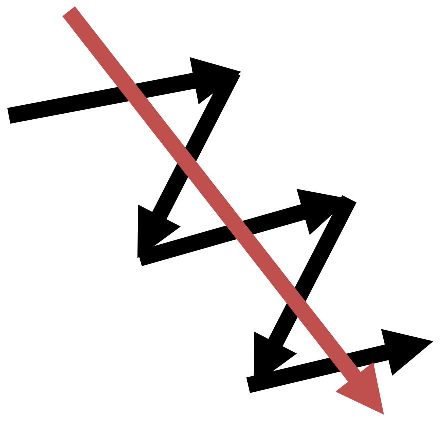
Gradient Descent Details

11×10^{-2} :oscillates
(Raw gradients)



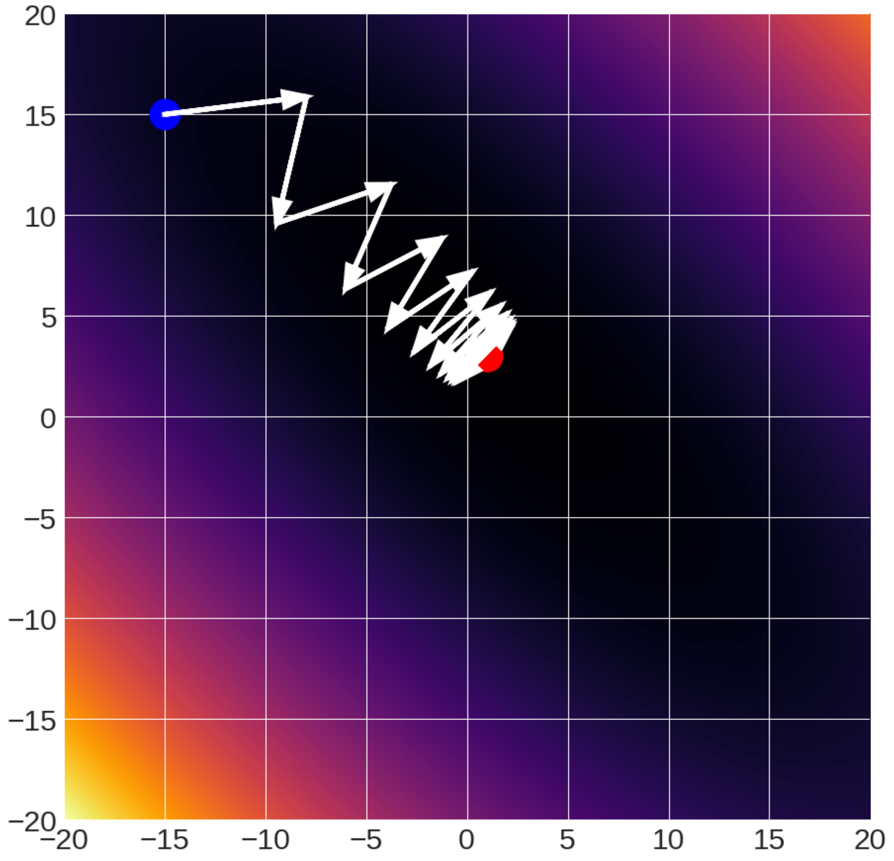
Solution:
Average gradients

With exponentially decaying weights, called "momentum"

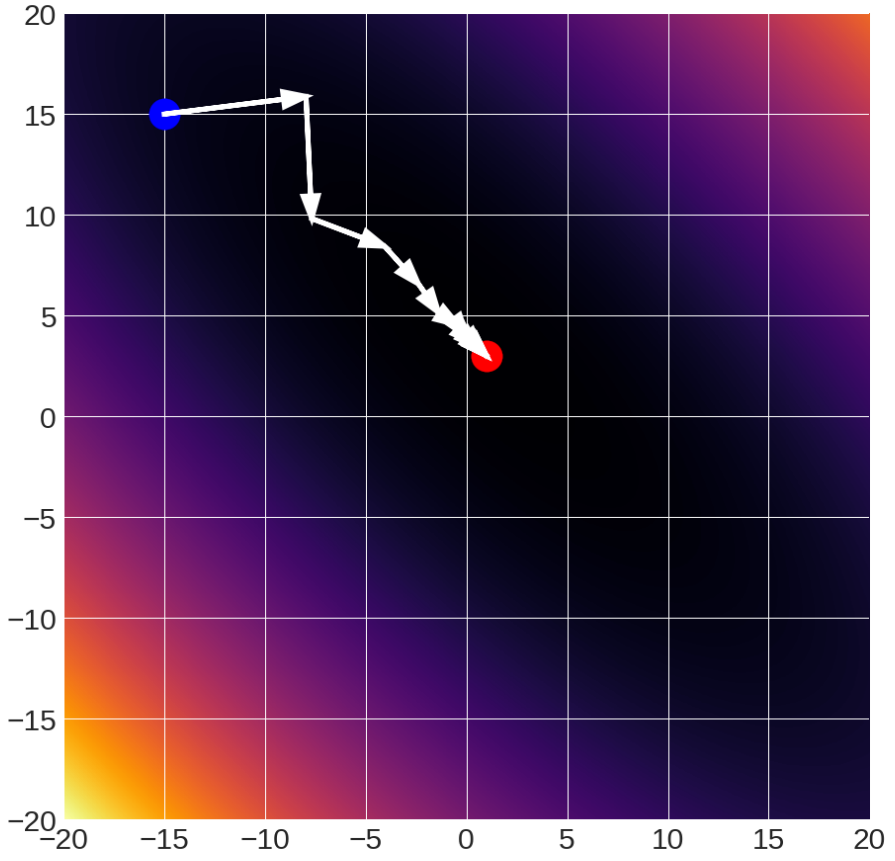


Gradient Descent Details

$11x10^{-2}$:oscillates
(Raw gradients)



$11x10^{-2}$
(0.25 momentum)

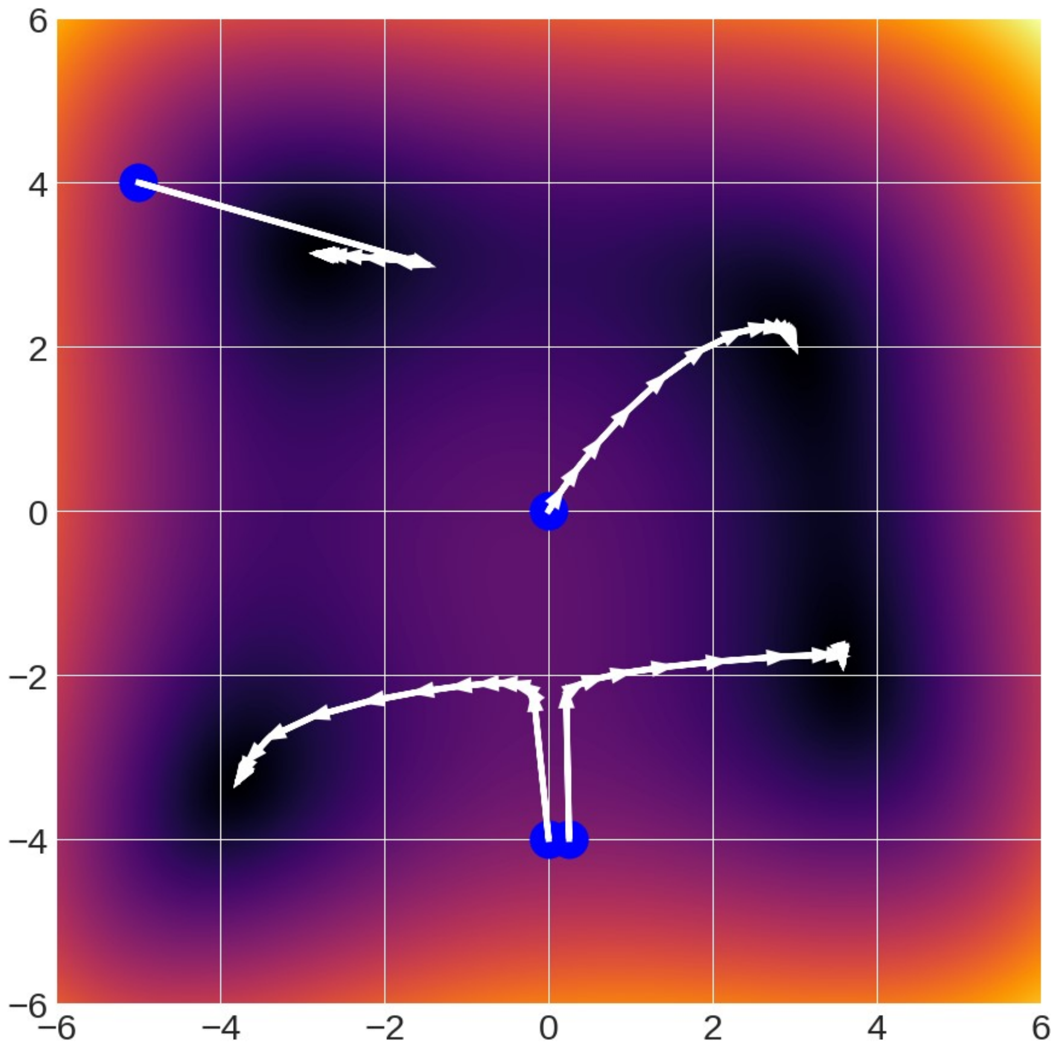


Gradient Descent Details

Multiple Minima

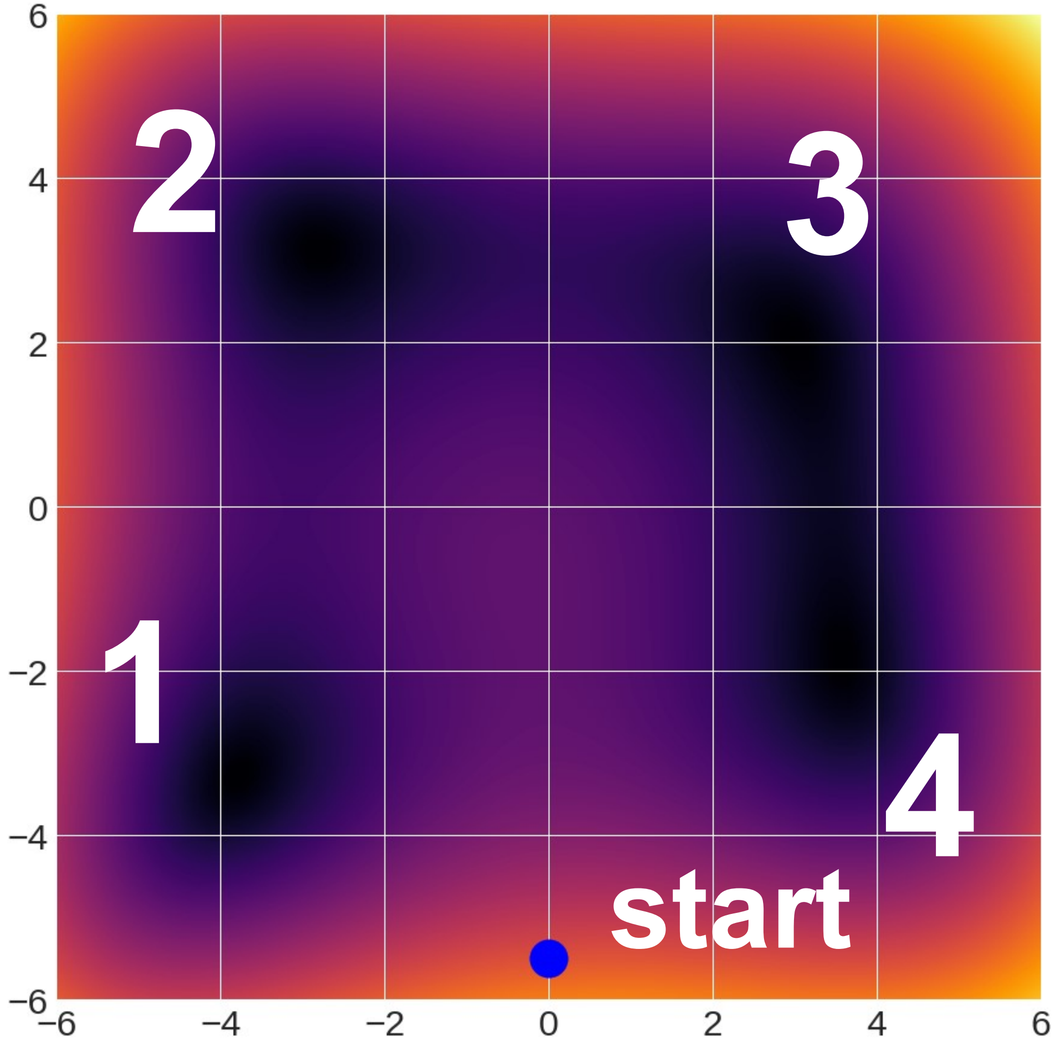
→

Gradient Descent Finds **local minimum**

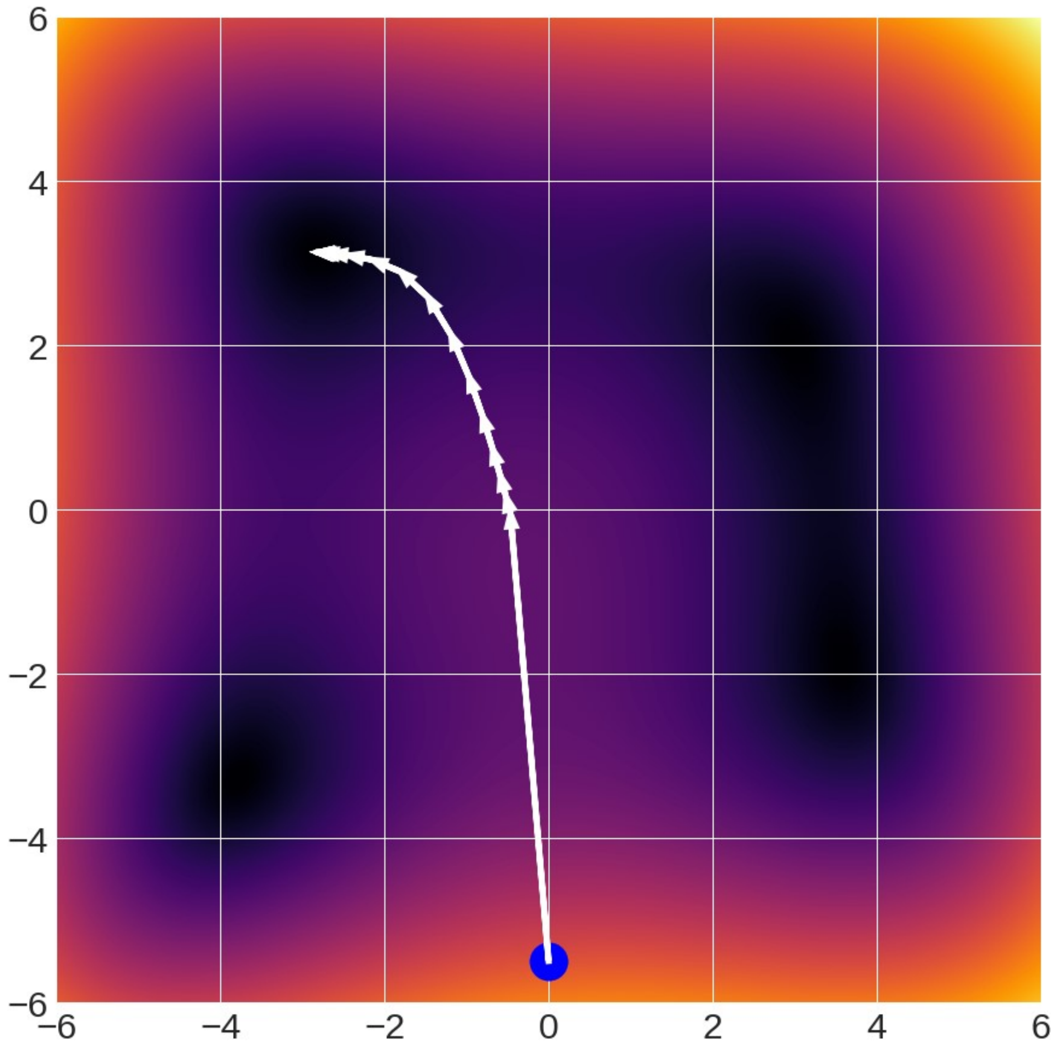


Gradient Descent Details

Guess the minimum!

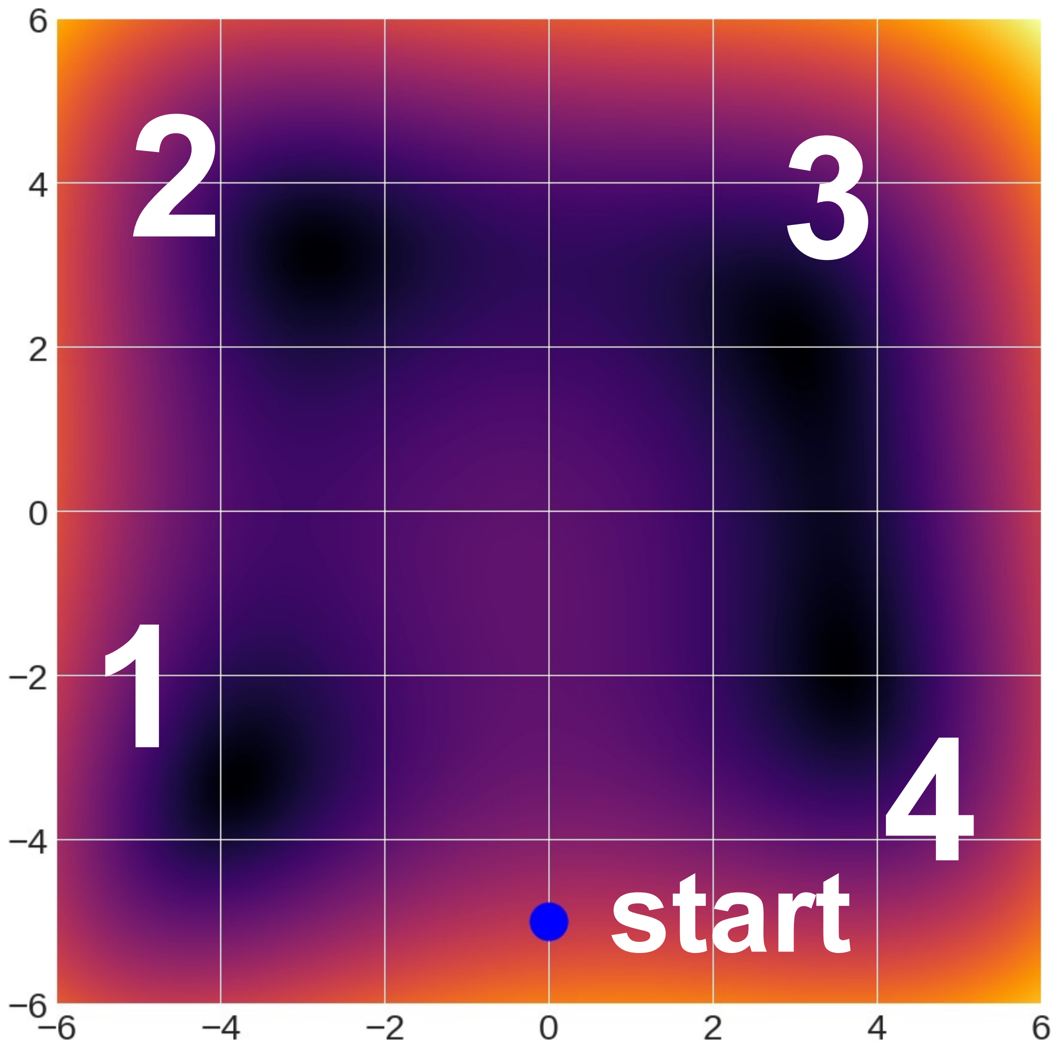


Gradient Descent Details



Gradient Descent Details

Guess the minimum!

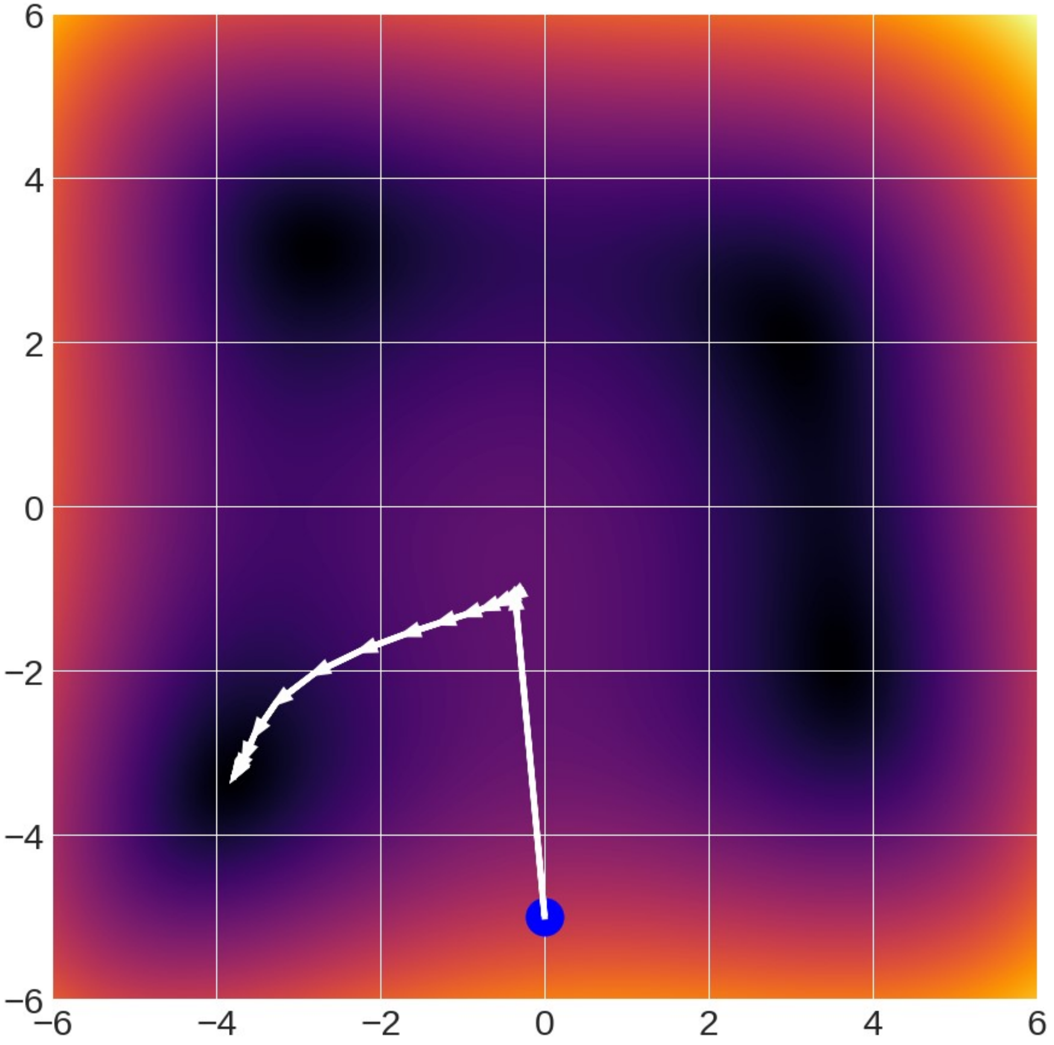


Gradient Descent Details

Dynamics are fairly complex

Many important functions are **convex**: any local minimum is a global minimum

Many important functions are not.



In practice

- **Conventional wisdom**: minibatch stochastic gradient descent (SGD) + momentum (package implements it for you) + some sensibly changing learning rate
- The above is typically what is meant by “SGD”
- Other update rules exist; benefits in general not clear (sometimes better, sometimes worse)

Optimizing Everything

$$L(\boldsymbol{W}) = \boldsymbol{\lambda}\|\boldsymbol{W}\|_{\boldsymbol{2}}^{\boldsymbol{2}} + \sum_{i=1}^{n} -\log\left(\frac{\exp((Wx)_{y_i})}{\sum_k \exp((Wx)_k))}\right)$$

$$L(\boldsymbol{w}) = \lambda\|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n}\left(y_i - \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{x}_{\boldsymbol{i}}\right)^2$$

- Optimize **w** on training set with SGD to maximize training accuracy
- Optimize λ with random/grid search to maximize validation accuracy
- Note: Optimizing λ on training sets it to 0

(Over/Under)fitting and Complexity

Let’s fit a polynomial: given x, predict y

Note: can do non-linear regression with copies of x

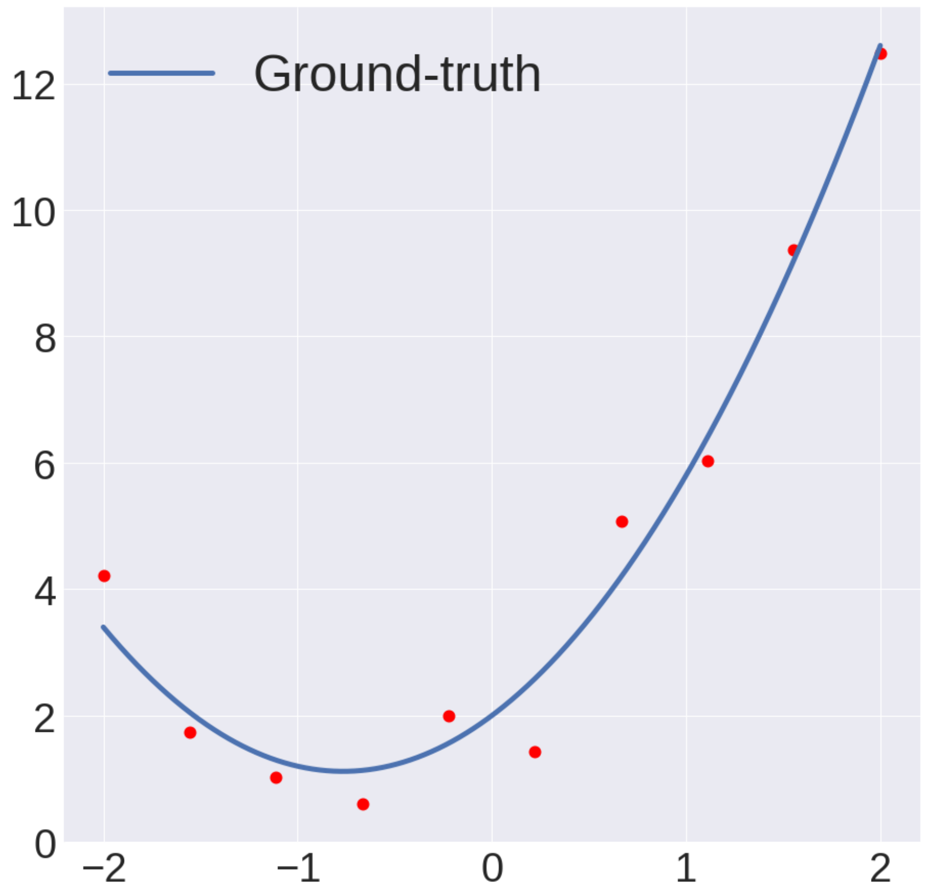
$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_1^F & \cdots & x_1^2 & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ x_N^F & \cdots & x_N^2 & x_N & 1 \end{bmatrix} \begin{bmatrix} w_F \\ \vdots \\ w_2 \\ w_1 \\ w_0 \end{bmatrix}$$

Matrix of all polynomial degrees

Weights: one per polynomial degree

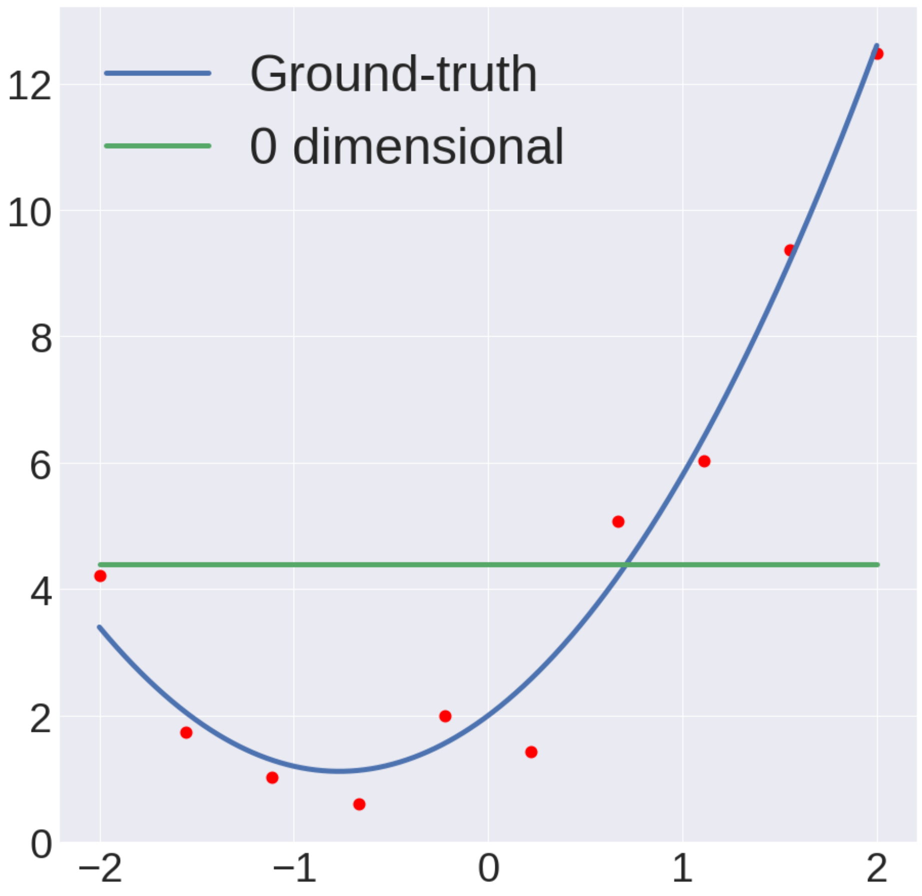
(Over/Under)fitting and Complexity

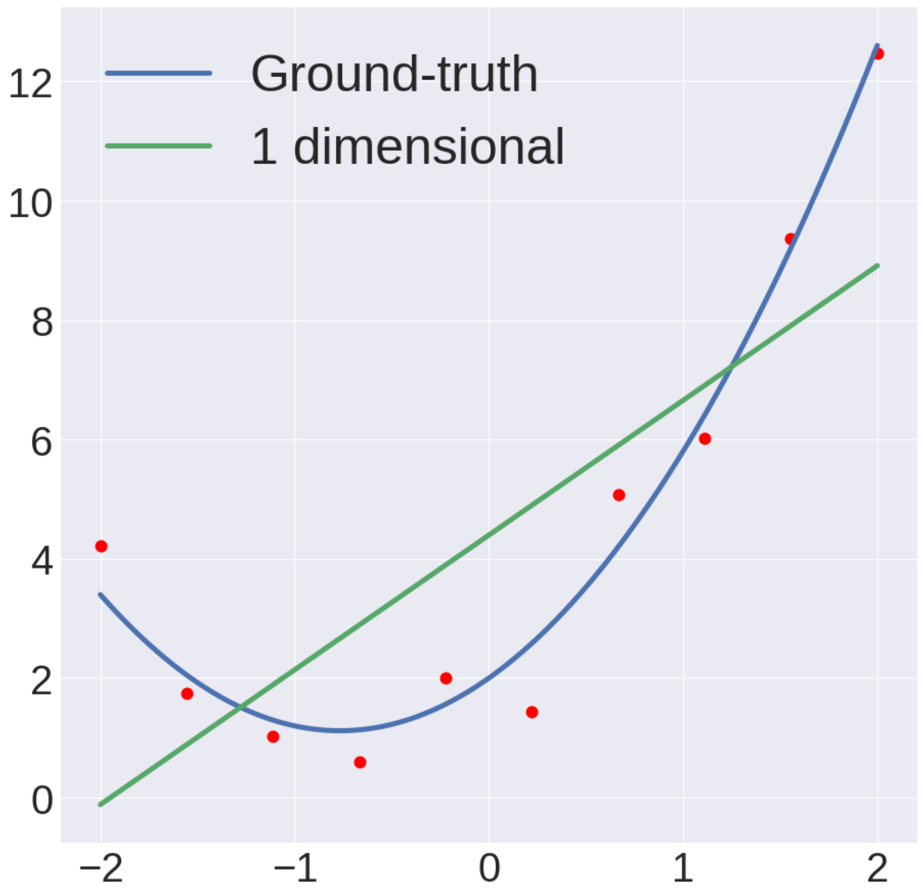
Model: $1.5x^2 + 2.3x+2 + N(0,0.5)$



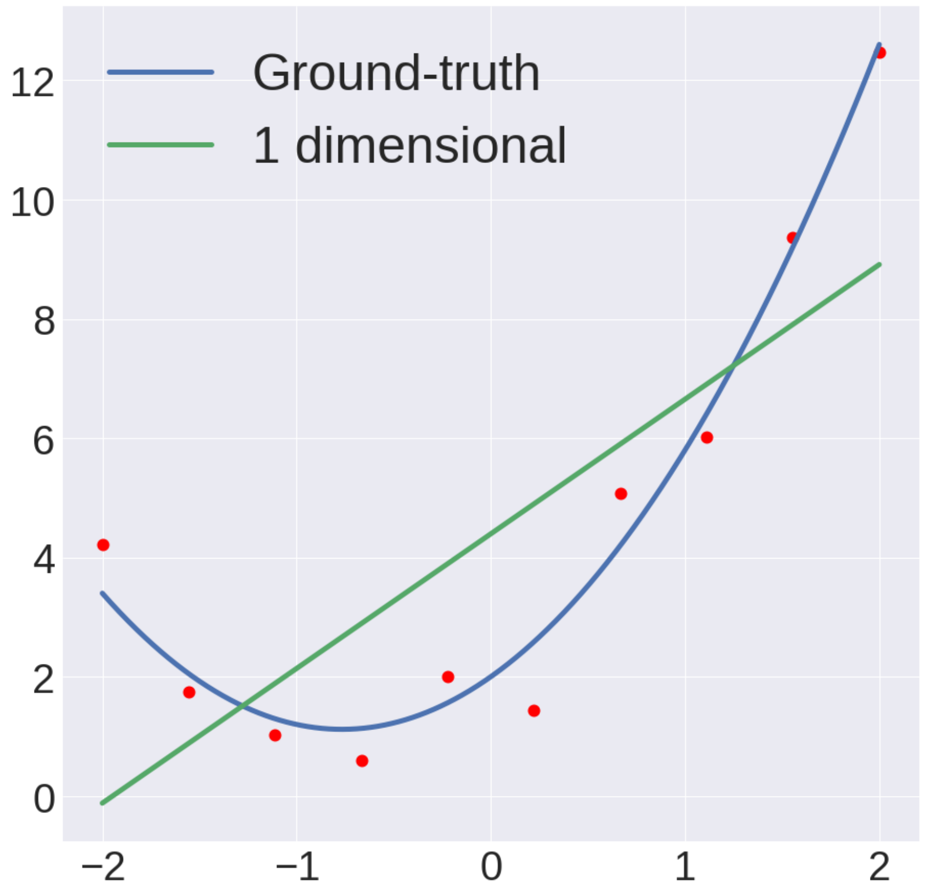
Underfitting

Model: $1.5x^2 + 2.3x+2 + N(0,0.5)$





Underfitting

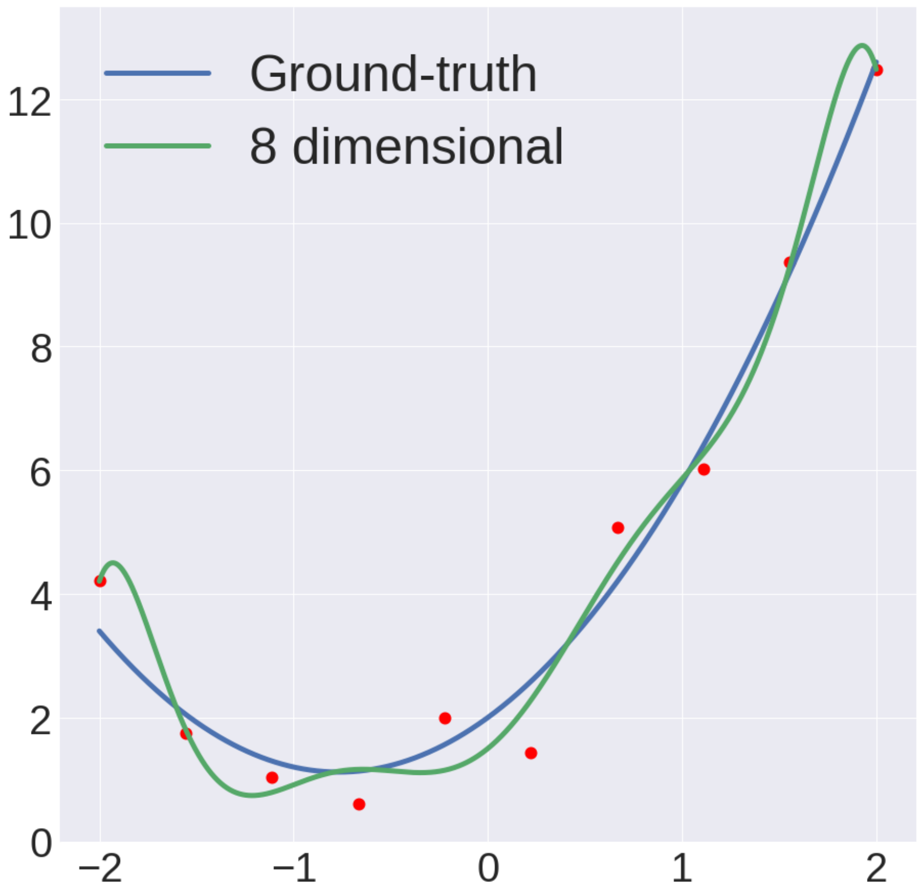


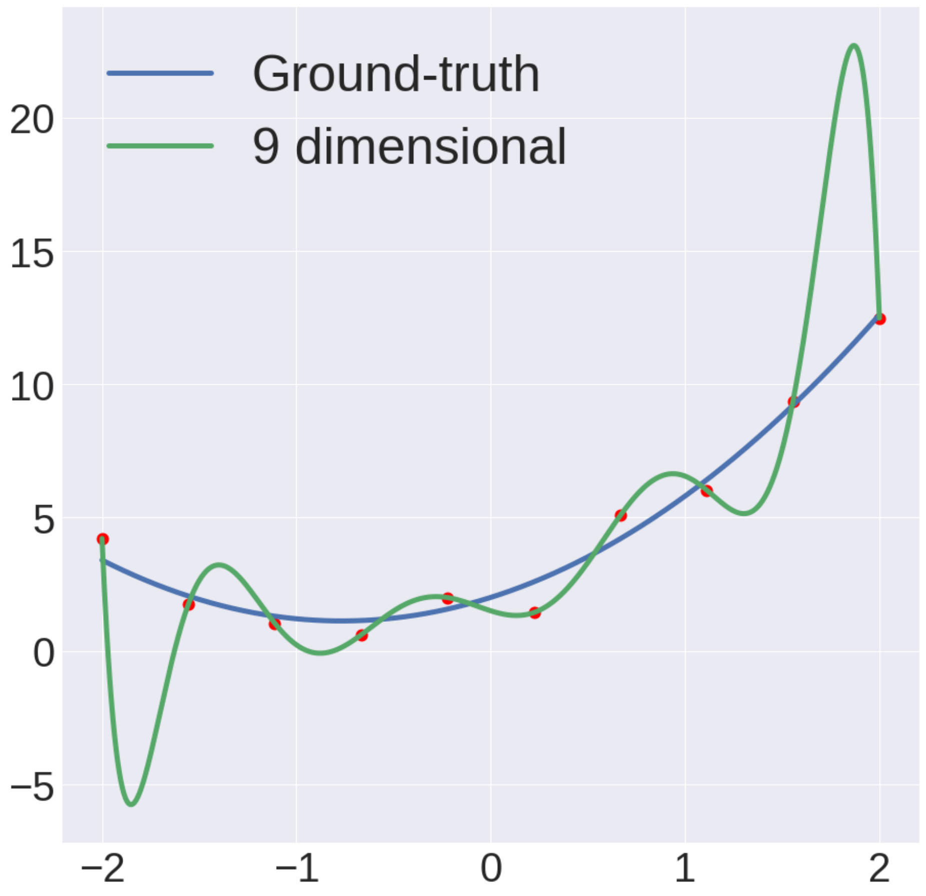
Model doesn’t have the parameters to fit the data.

Bias (statistics): Error intrinsic to the model.

Overfitting

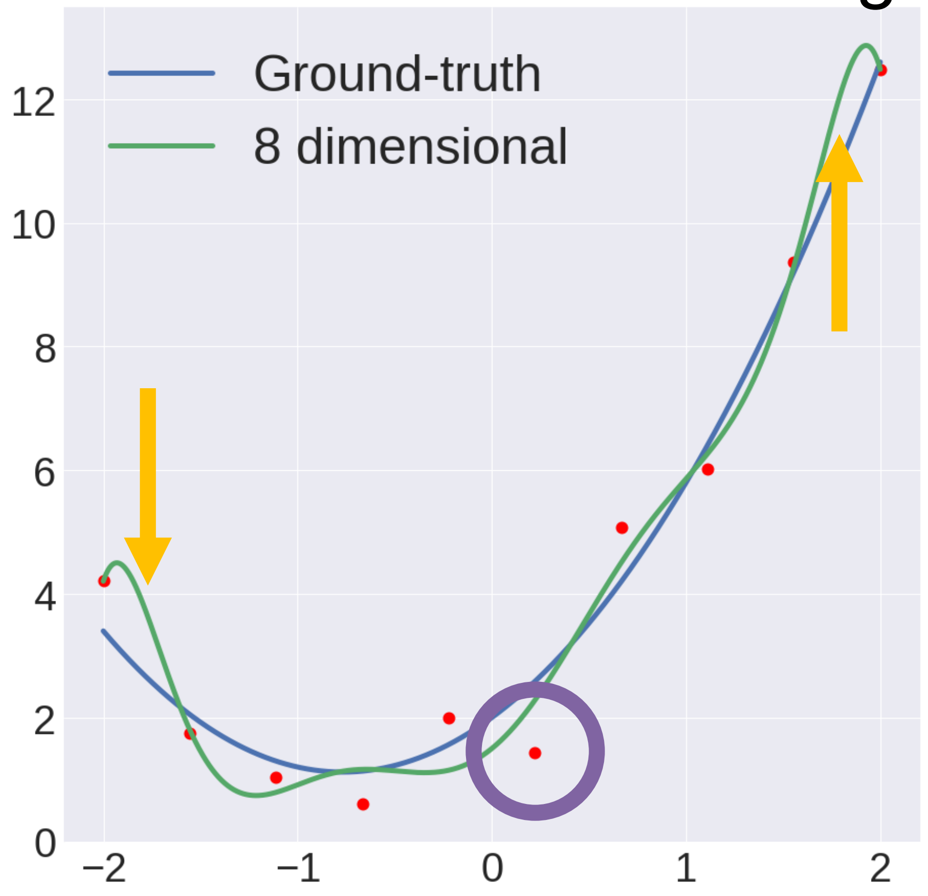
Model: $1.5x^2 + 2.3x+2 + N(0,0.5)$

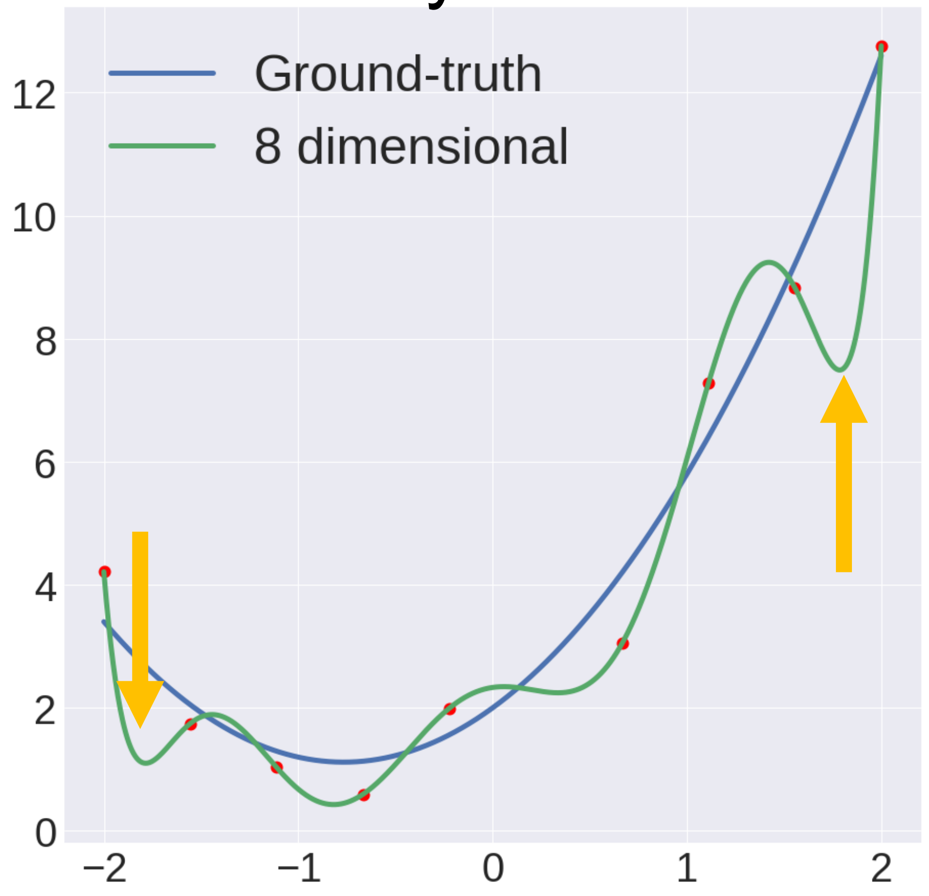




Overfitting

Model has high ***variance***: remove **one point**, and model changes dramatically





(Continuous) Model Complexity

$$\arg\min_{W} \lambda \|W\|_2^2 + \underbrace{\sum_{i=1}^{n} -\log\left(\frac{\exp((Wx)_{y_i})}{\sum_k \exp((Wx)_k)}\right)}$$

Regularization: penalty for complex model

Pay penalty for negative log-likelihood of correct class

Intuitively: big weights = more complex model

Model 1: $0.01*x_1 + 1.3*x_2 + -0.02*x_3 + -2.1x_4 + 10$

Model 2: $37.2*x_1 + 13.4*x_2 + 5.6*x_3 + -6.1x_4 + 30$

Fitting Model

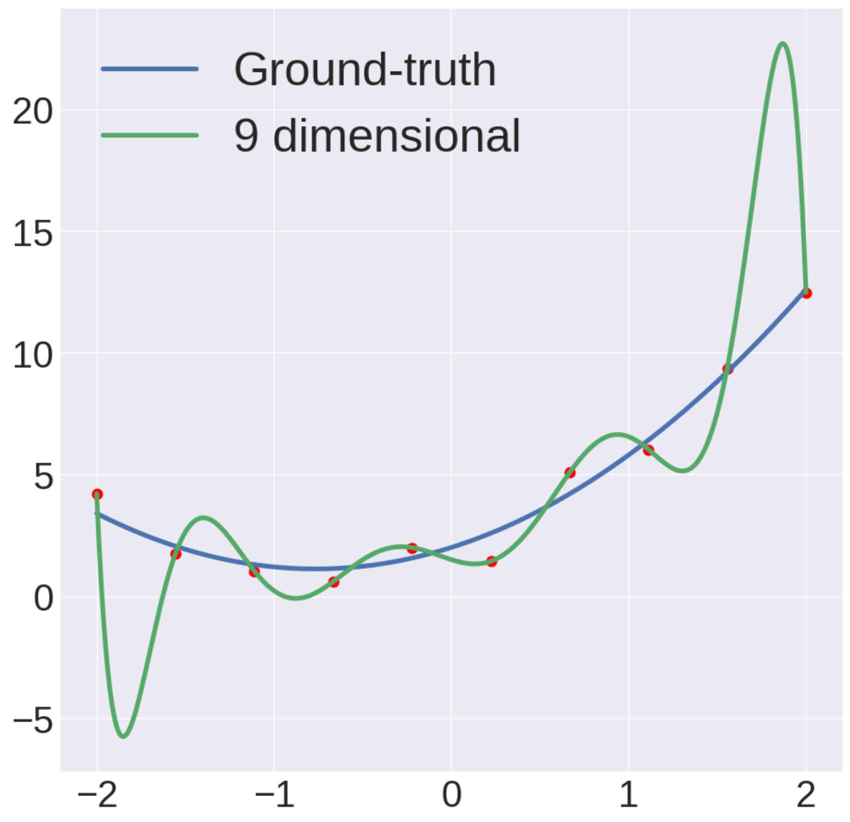
Again, fitting polynomial, but with regularization

$$\arg\min_{\mathrm{w}} ||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}|| + \lambda ||\boldsymbol{w}||$$

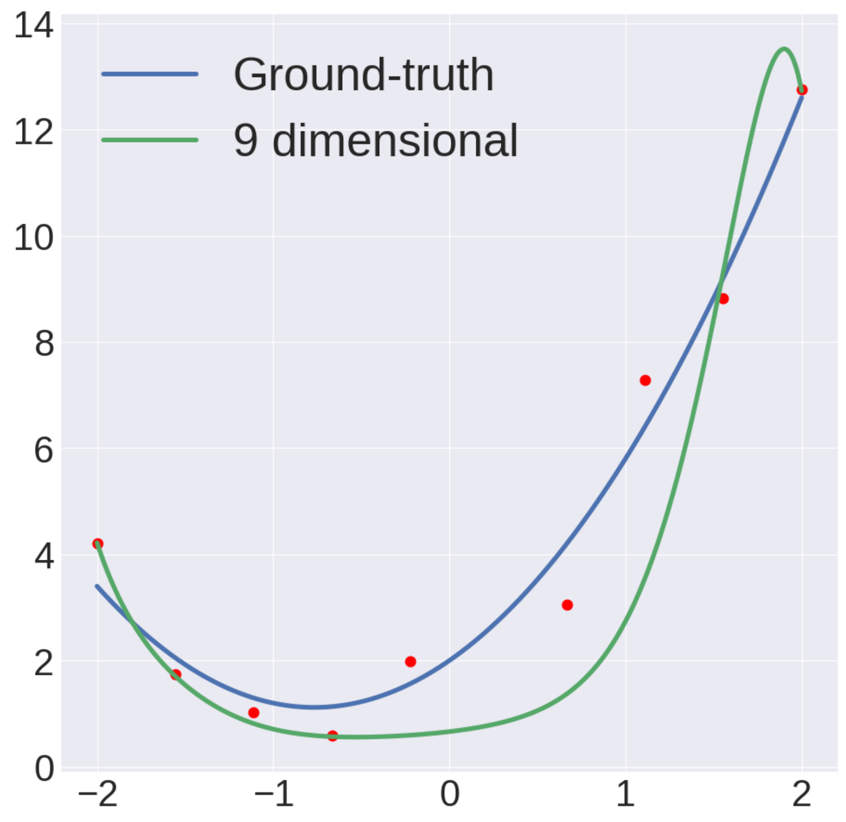
$$\begin{bmatrix} x_1^F & \cdots & x_1^2 & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ x_N^F & \cdots & x_N^2 & x_N & 1 \end{bmatrix} \quad \begin{bmatrix} w_F \\ \vdots \\ w_0 \end{bmatrix}$$

Adding Regularization

No regularization:
fits all data points



Regularization:
can't fit all data points



In General

Error on new data comes from combination of:

1. Bias: model is oversimplified and can’t fit the underlying data
2. Variance: you don’t have the ability to estimate your model from limited data
3. Inherent: the data is intrinsically difficult

Bias and variance trade-off. Fixing one hurts the other.

Underfitting and Overfitting

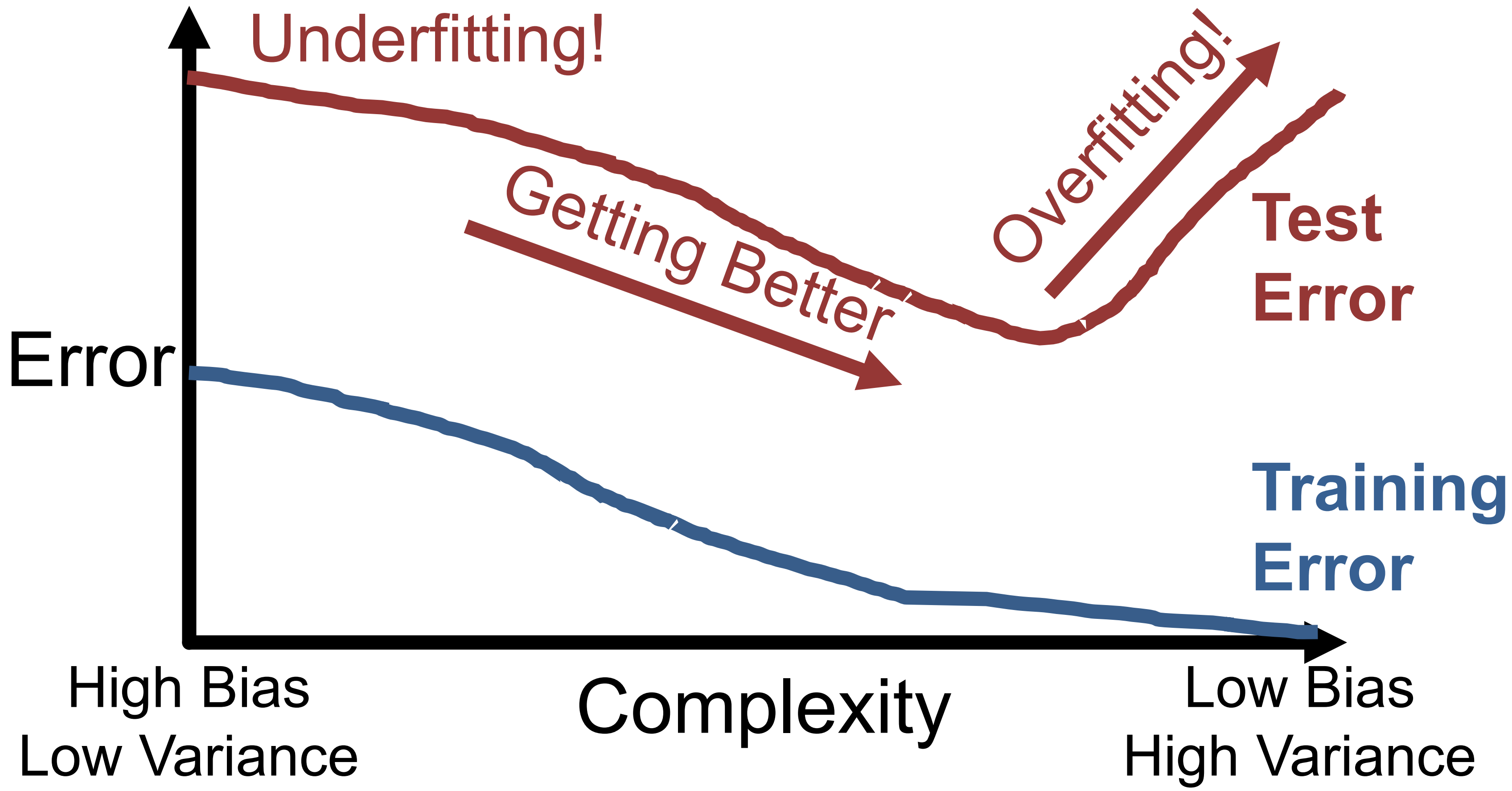


Diagram adapted from: D. Hoiem

Underfitting and Overfitting

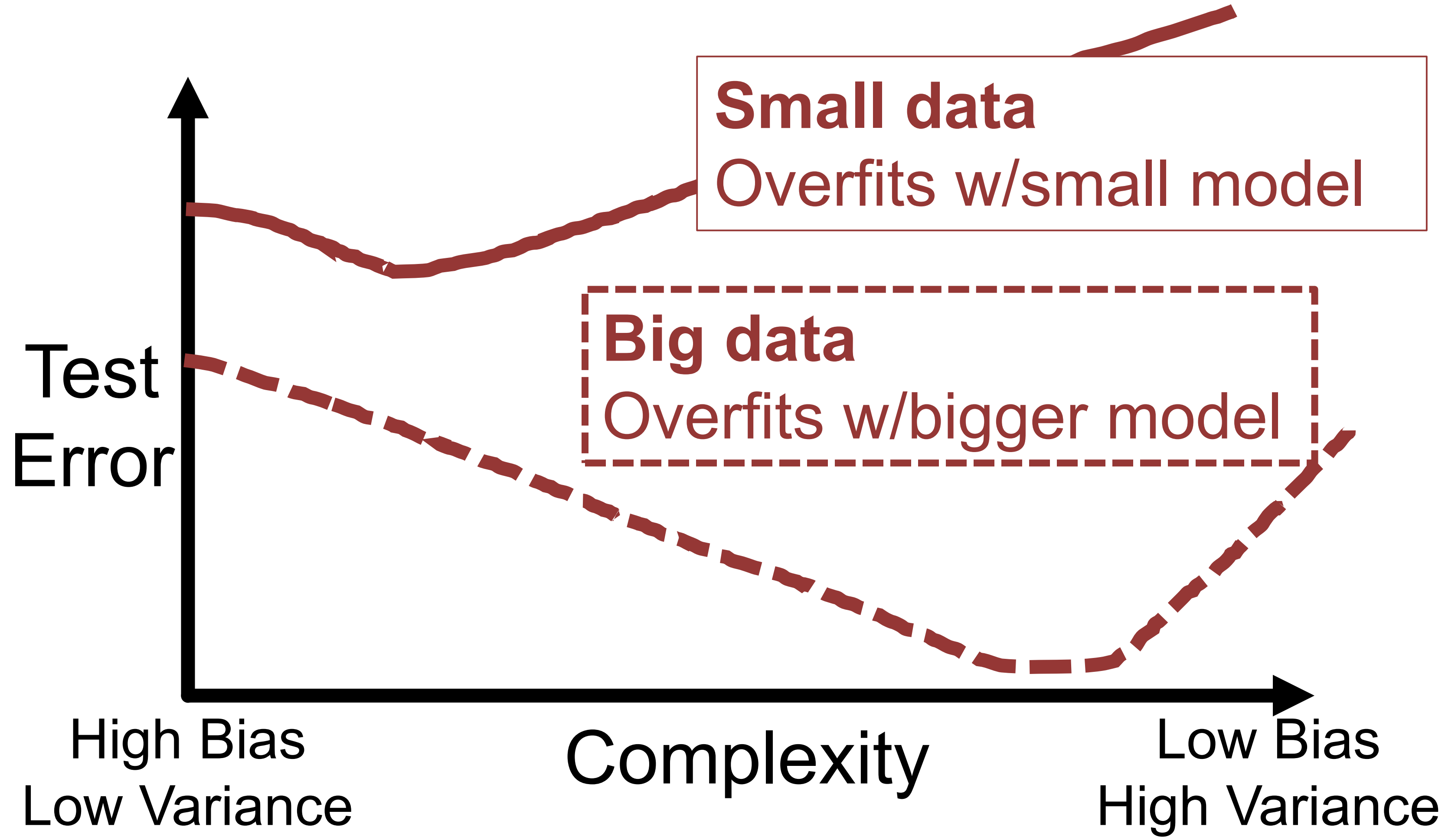
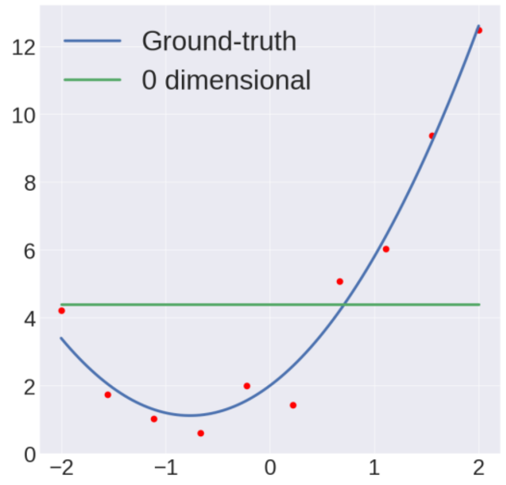
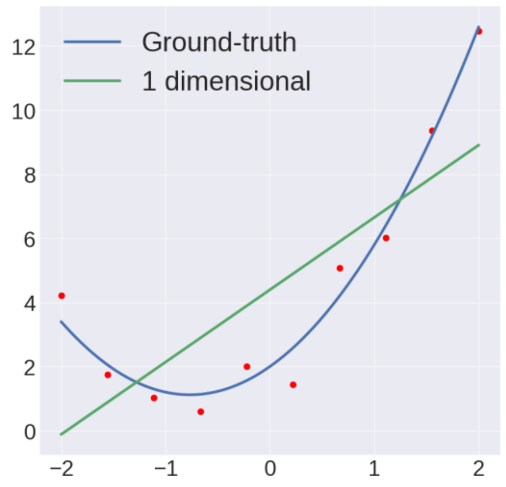


Diagram adapted from: D. Hoiem

Underfitting



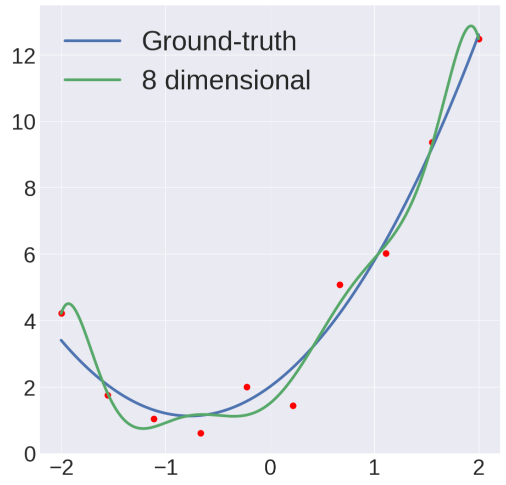


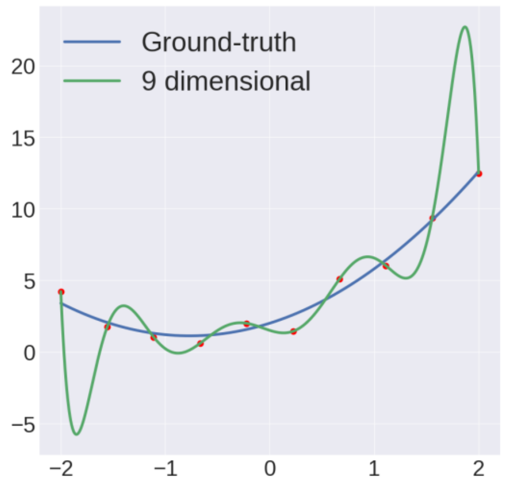
Do poorly on both training and validation data due to bias.

Solution:

1. More features
2. More powerful model
3. Reduce regularization

Overfitting





Do well on training data, but poorly on validation data due to variance

Solution:

1. More data
2. Less powerful model
3. Regularize your model more

Cris Dima rule: first make sure you *can* overfit, then stop overfitting.

Next Class

- Non-linear models (neural nets)

Let’s Compute Another Gradient

- Below is another derivation that’s worth looking at on your own time if you’re curious

Computing The Gradient

Multiclass Support Vector Machine

$$\arg\min_{\boldsymbol{W}} \lambda \|\boldsymbol{W}\|_2^2 + \sum_{i=1}^{n} \sum_{j \neq y_i} \max(0, (\boldsymbol{W}\boldsymbol{x}_i)_j - (\boldsymbol{W}\boldsymbol{x}_i)_{y_i} + m)$$

Notation:

W → rows w_i (i.e., per-class scorer)

$(Wx_i)_j \rightarrow w_j^T x_i$

$$\arg\min_{\boldsymbol{W}} \lambda \sum_{j=1}^{K} \|\boldsymbol{w}_j\|_2^2 + \sum_{i=1}^{n} \sum_{j \neq y_i} \max(0, \boldsymbol{w}_j^T \boldsymbol{x}_i - \boldsymbol{w}_{y_i}^T \boldsymbol{x}_i + m)$$

Derivation setup: Karpathy and Fei-Fei

Computing The Gradient

$$\arg\min_{\boldsymbol{W}} \boldsymbol{\lambda} \sum_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{K}} \|\boldsymbol{w}_{\boldsymbol{j}}\|_{\mathbf{2}}^{\mathbf{2}} + \sum_{i=1}^{n} \underbrace{\sum_{j \neq y_i} \max(0, \boldsymbol{w}_j^T \boldsymbol{x}_i - \boldsymbol{w}_{y_i}^T \boldsymbol{x}_i + m)}$$

$$\frac{\partial}{\partial w_j}: \quad \begin{array}{ll} w_j^T x_i - w_{y_i}^T x_i + m \leq 0: & 0 \\ w_j^T x_i - w_{y_i}^T x_i + m > 0: & x_i \end{array}$$

$$\rightarrow 1(w_j^T x_i - w_{y_i}^T x_i + m > 0)x_i$$

Derivation setup: Karpathy and Fei-Fei

Computing The Gradient

$$\arg\min_{\boldsymbol{W}} \boldsymbol{\lambda} \sum_{j=1}^{K} \left\|\boldsymbol{w}_j\right\|_2^2 + \sum_{i=1}^{n} \underbrace{\sum_{j \neq y_i} \max(0, \boldsymbol{w}_j^T \boldsymbol{x}_i - \boldsymbol{w}_{y_i}^T \boldsymbol{x}_i + m)}$$

$$\frac{\partial}{\partial w_{y_i}} : \quad \sum_{j \neq y_i} 1\left(w_j^T x_i - w_{y_i}^T x_i + m > 0\right)(-x_i)$$

Derivation setup: Karpathy and Fei-Fei

Interpreting The Gradient

$$-\frac{\partial}{\partial w_j}: \underbrace{1(w_j^T x_i - w_{y_i}^T x_i + m > 0)}\ \underbrace{-x_i}$$

If we do not predict the correct class by at least a score difference of m …

Want incorrect class's scoring vector to score that point lower.

Recall:
Before: $w^T x$;
After: $(w-\alpha x)^T x = w^T x - a x^T x$

Derivation setup: Karpathy and Fei-Fei