Numerical Linear Algebra

EECS 442 – David Fouhey

Fall 2019, University of Michigan

http://web.eecs.umich.edu/~fouhey/teaching/EECS442_W19/

Administrivia

- HW 1 out – due in two weeks
- Follow submission format (wrong format = 0)
- The homeworks are **not** fill-in-the-blank. This is harder to do but mirrors life
- If it's ambiguous: make a decision, document what you think and why in your homework, and move on
- Highly encouraged to work together. See piazza
- Please check syllabus for what's allowed. I guarantee checking the syllabus thoroughly will help boost your grade.

This Week – Math

Two goals for the next two classes:

- Math with computers ≠ Math
- Practical math you need to know but may not have been taught

This Week – Goal

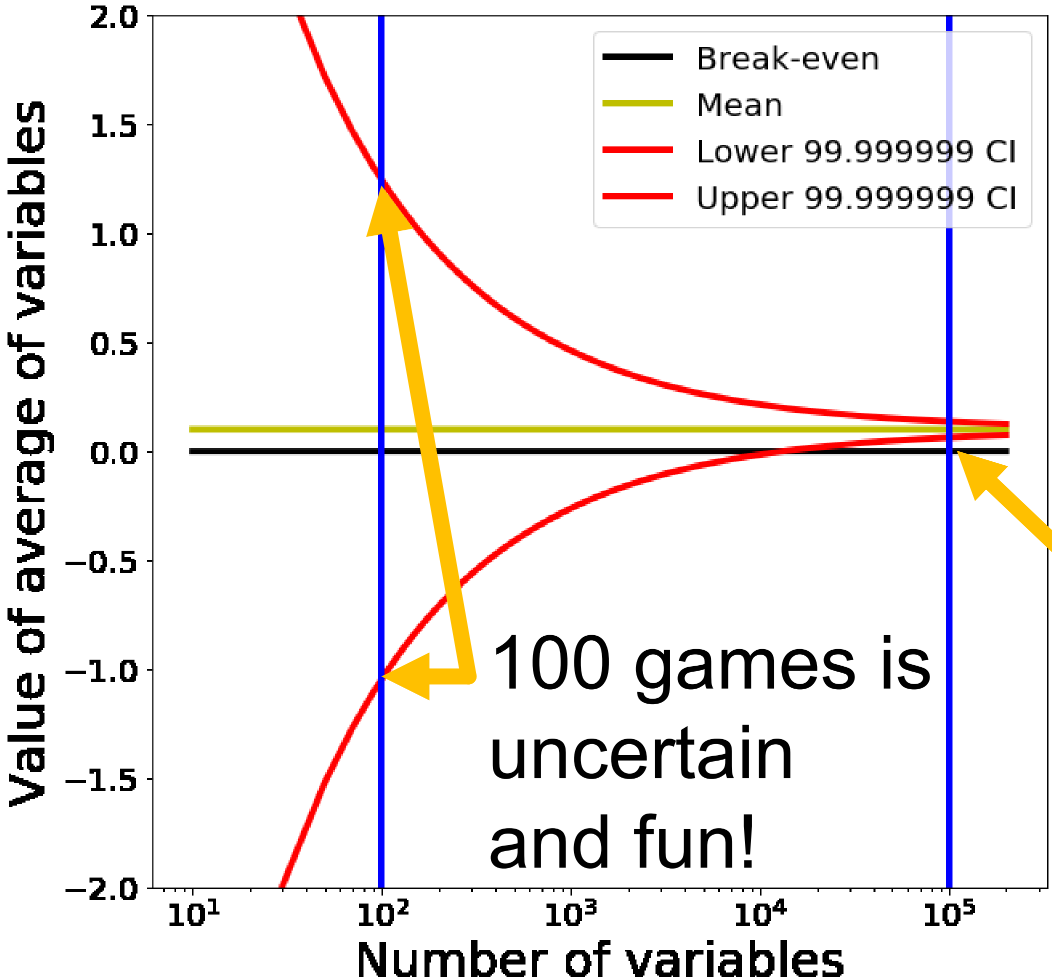
- Not a “Linear algebra in two lectures” – that’s impossible.
- *Some of this you should know!*
- Aimed at reviving your knowledge and plugging any gaps
- Aimed at giving you intuitions

Adding Numbers

- **1 + 1 = ?**
- Suppose x_i is normally distributed with mean μ and standard deviation σ for $1 \leq i \leq N$
- **How is the average, or $\boldsymbol{\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i}$, distributed (qualitatively), in terms of variance?**
- *The Free Drinks in Vegas Theorem*: $\hat{\mu}$ has mean μ and standard deviation $\frac{\sigma}{\sqrt{N}}$.

Free Drinks in Vegas

Each game/variable has mean $0.10, std $2



100K games is guaranteed profit: 99.999999% lowest value is $0.064. $0.01 for drinks $0.054 for profits

Let’s Make It Big

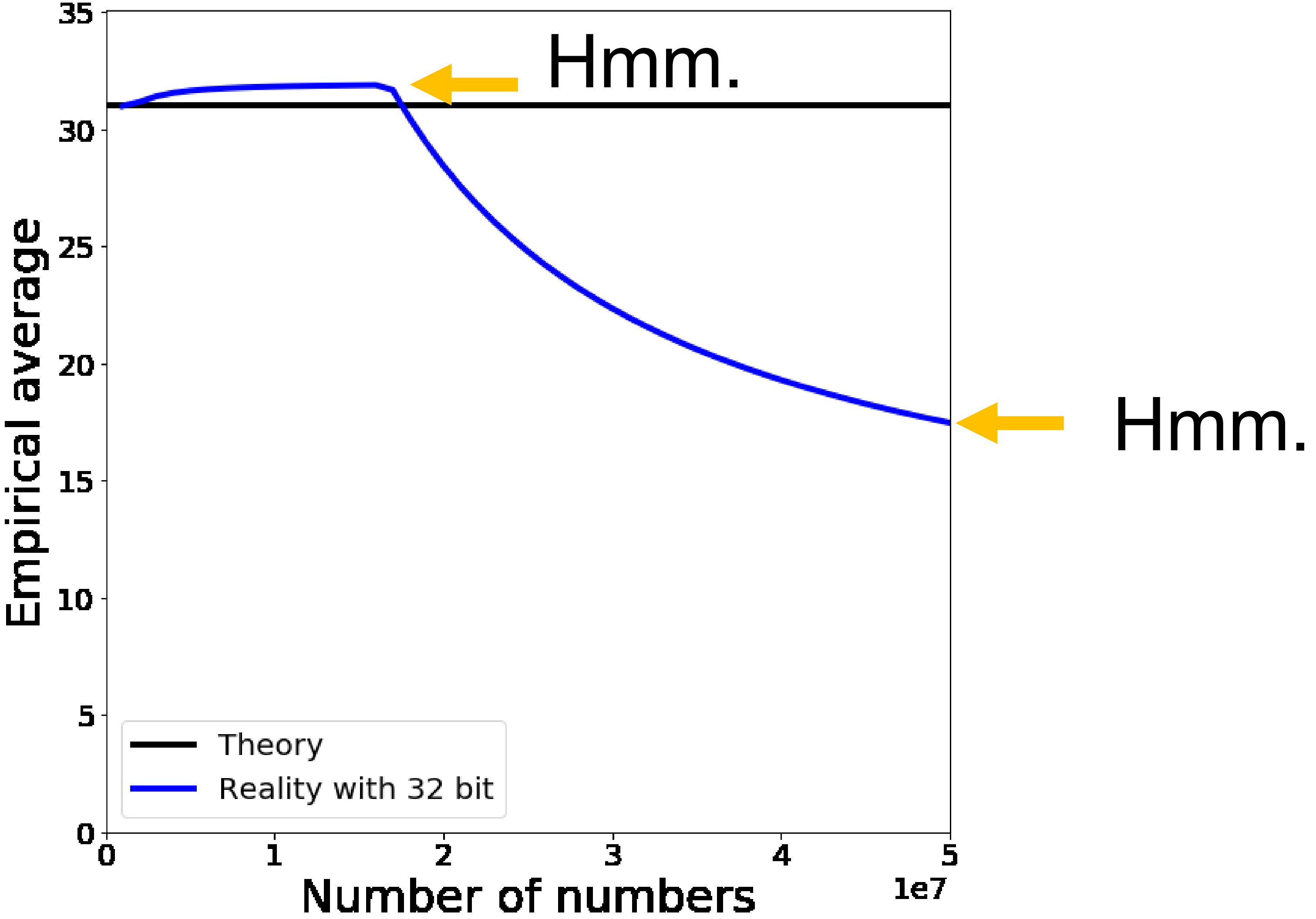
- Suppose I average 50M normally distributed numbers (mean: 31, standard deviation: 1)
- For instance: have predicted and actual depth for 200 480x640 images and want to know the average error (|predicted – actual|)

```
numerator = 0
for x in xs:
  numerator += x
return numerator / len(xs)
```

Let's Make It Big

- **What should happen qualitatively?**
- Theory says that the average is distributed with mean 31 and standard deviation $\frac{1}{\sqrt{50M}} \approx (10^{-5})$
- **What will happen?**
- Reality: 17.47

Trying it Out



What's a Number?

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	0	1	1	1	0	0	1	185

128 + 32 + 16 + 8 + 1 = 185

Adding Two Numbers

	2^8		2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
			1	0	1	1	1	0	0	1	185
		+	0	1	1	0	1	0	0	1	105
	1		0	0	1	0	0	0	1	0	34
	Carry Flag		Result								

"Integers" on a computer are integers <u>*modulo 2^k*</u>

Some Gotchas

32 + (3 / 4) x 40 = 32

32 + (3 x 40) / 4 = 62

Why?

Underflow	No Underflow
32 + 3 / 4 x 40 =	32 + 3 x 40 / 4 =
32 + 0 x 40 =	32 + 120 / 4 =
32 + 0 =	32 + 30 =
32	62

Ok – you have to multiply before dividing

Some Gotchas

Should be: 9x4=36

math 32 + (9 x 40) / 10 = 68

uint8 32 + (9 x 40) / 10 = 42

Overflow

32 + 9 x 40 / 10 =
32 + 104 / 10 =
32 + 10 =
42

Why 104?

9 x 40 = 360
360 % 256 = 104

What’s a Number?

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	0	1	1	1	0	0	1	185

How can we do fractions?

2^5	2^4	2^3	2^2	2^1	2^0	•	2^{-1}	2^{-2}	
1	0	1	1	1	0	•	0	1	45.25

45 0.25

Fixed-Point Arithmetic

2^5	2^4	2^3	2^2	2^1	2^0 ●	2^{-1}	2^{-2}	
1	0	1	1	1	0	0	1	45.25

What's the largest number we can represent?

63.75 – **Why?**

How precisely can we measure at 63?

0.25

How precisely can we measure at 0?

0.25

Fine for many purposes but for science, seems silly

Floating Point

Sign (S)	Exponent (E)				Fraction (F)		
1	0	1	1	1	0	0	1

1 | **7** | **1**

-1 | $\mathbf{2^{7-7} = 2^0 = 1}$ | **1+1/8 = 1.125**

$$\left(-1^{S}\right)\left(2^{E+bias}\right)\left(1+\frac{F}{2^3}\right)$$

Bias: allows exponent to be negative; Note: fraction = significant = mantissa; exponents of all ones or all zeros are special numbers

Floating Point

Sign	Exponent		Fraction	
1	0 1 1 1	0/8	0 0 0	$-2^0 \times 1.00 = -1$
		1/8	0 0 1	$-2^0 \times 1.125 = -1.125$
		2/8	0 1 0	$-2^0 \times 1.25 = -1.25$
			...	
		6/8	1 1 0	$-2^0 \times 1.75 = -1.75$
		7/8	1 1 1	$-2^0 \times 1.875 = -1.875$
-1	7-7=0 *(-bias)*			

Floating Point

Sign	Exponent
1	1 0 0 1
-1	9-7=2 *(-bias)*

Fraction

	Fraction	
0/8	0 0 0	$-2^2 \times 1.00 = -4$
1/8	0 0 1	$-2^2 \times 1.125 = -4.5$
2/8	0 1 0	$-2^2 \times 1.25 = -5$
	...	
6/8	1 1 0	$-2^2 \times 1.75 = -7$
7/8	1 1 1	$-2^2 \times 1.875 = -7.5$

Floating Point

Sign	Exponent	Fraction	
1	0 1 1 1 1	0 0 0	-2^0 x 1.00 = -1
		0 0 1	-2^0 x 1.125 = -1.125
1	1 0 0 1	0 0 0	-2^2 x 1.00 = -4
		0 0 1	-2^2 x 1.125 = -4.5

Gap between numbers is *relative*, not absolute

Revisiting Adding Numbers

	Sign	Exponent	Fraction	
	1	0 1 1 0	0 0 0	$-2^{-1} \times 1.00 = -0.5$
+	1	1 0 0 1	0 0 0	$-2^{2} \times 1.00 = -4$
	1	1 0 0 1	0 0 1	$-2^{2} \times 1.125 = -4.5$

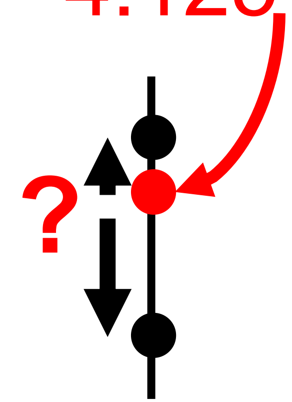
Actual implementation is complex

Revisiting Adding Numbers

	Sign	Exponent	Fraction	
	1	0 1 0 0	0 0 0	$-2^{-3} \times 1.00 = -0.125$
+	1	1 0 0 1	0 0 0	$-2^{2} \times 1.00 = -4$

$-2^{2} \times 1.03125 = -4.125$

Sign	Exponent	Fraction	
1	1 0 0 1	0 0 0	$-2^{2} \times 1.00 = -4$
1	1 0 0 1	0 0 1	$-2^{2} \times 1.125 = -4.5$

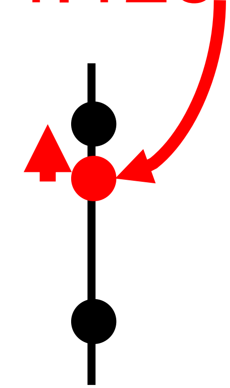


Revisiting Adding Numbers

	Sign	Exponent	Fraction	
	1	0 1 0 0	0 0 0	$-2^{-3} \times 1.00 = -0.125$
+	1	1 0 0 1	0 0 0	$-2^{2} \times 1.00 = -4$

$-2^{2} \times 1.03125 = -4.125$

Sign	Exponent	Fraction	
1	1 0 0 1	0 0 0	$-2^{2} \times 1.00 = -4$



For a and b, these can happen

a + b = a   a+b-a ≠ b

Revisiting Adding Numbers

IEEE 754 Single Precision / Single

8 bits

$2^{127} \approx 10^{38}$

23 bits

$\approx$ 7 decimal digits

IEEE 754 Double Precision / Double

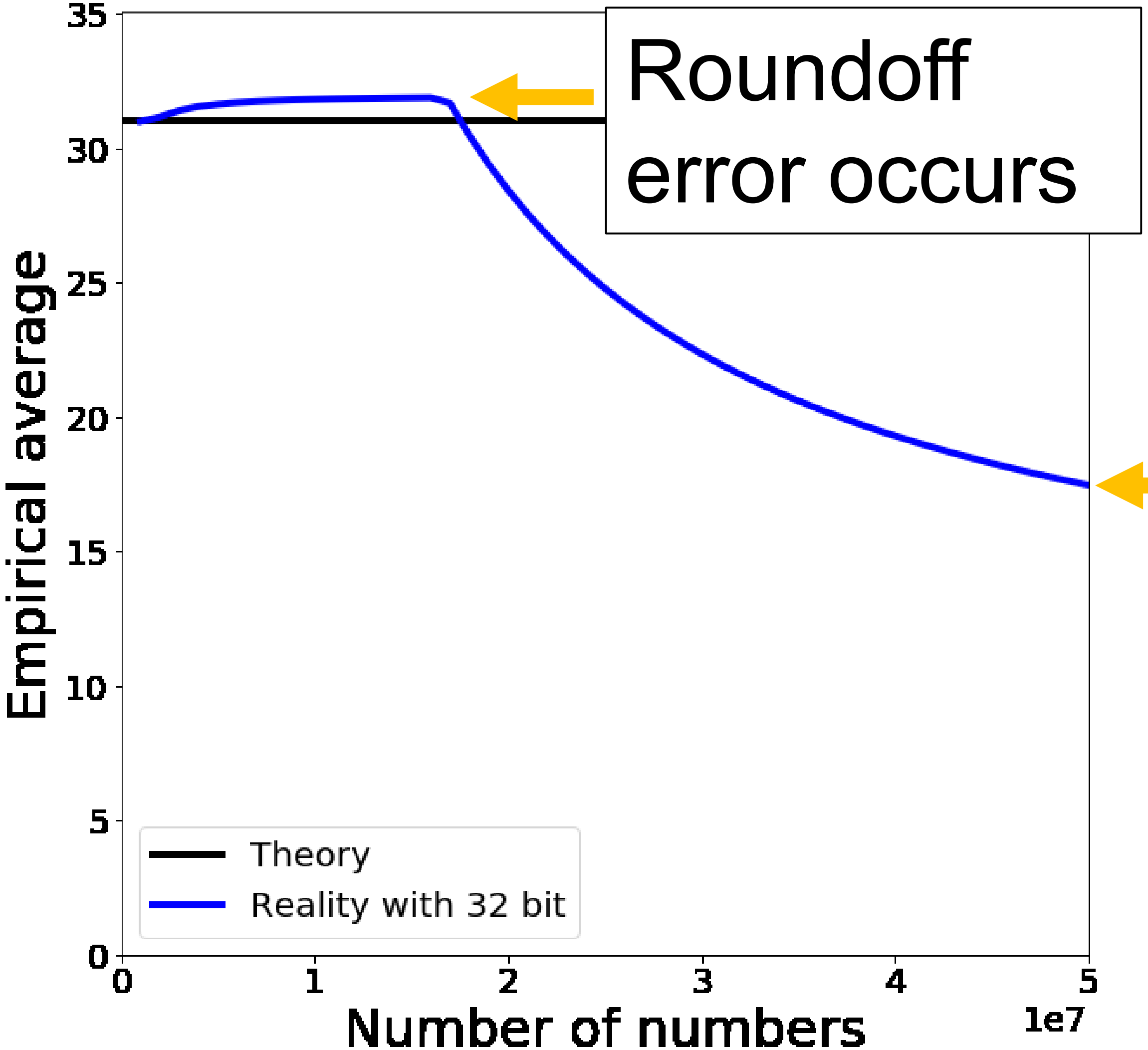
11 bits

$2^{1023} \approx 10^{308}$

52 bits

$\approx$ 15 decimal digits

Trying it Out



a+b=a -> numerator is stuck, denominator isn’t

Take-homes

- Computer numbers aren’t math numbers
- Overflow, accidental zeros, roundoff error, and basic equalities are almost certainly incorrect for some values
- Floating point defaults and numpy try to protect you.
- Generally safe to use a double and use built-in-functions in numpy (not necessarily others!)
- Spooky behavior = look for numerical issues

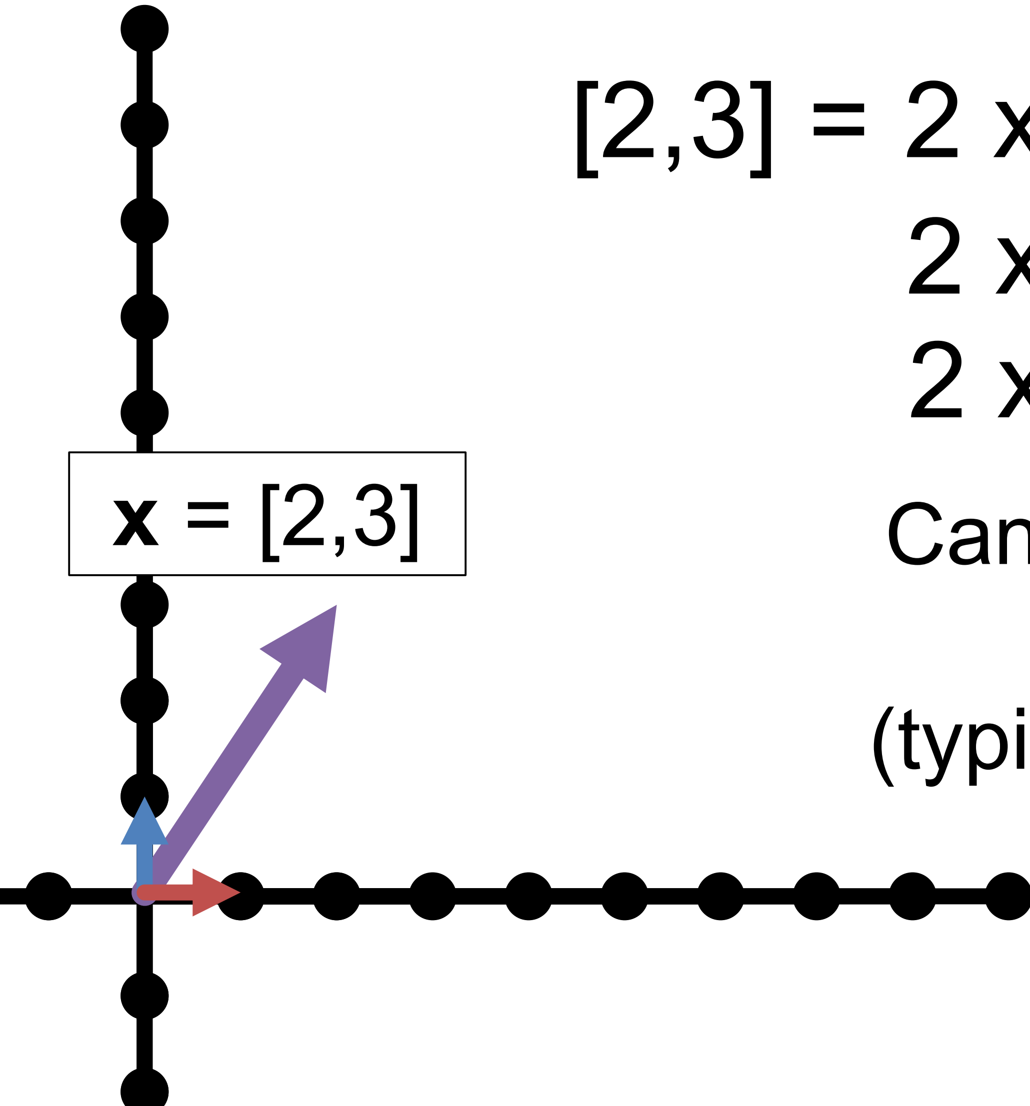
Vectors

$$[2,3] = 2 \times [1,0] + 3 \times [0,1]$$

$$2 \times \rightarrow + 3 \times \uparrow$$

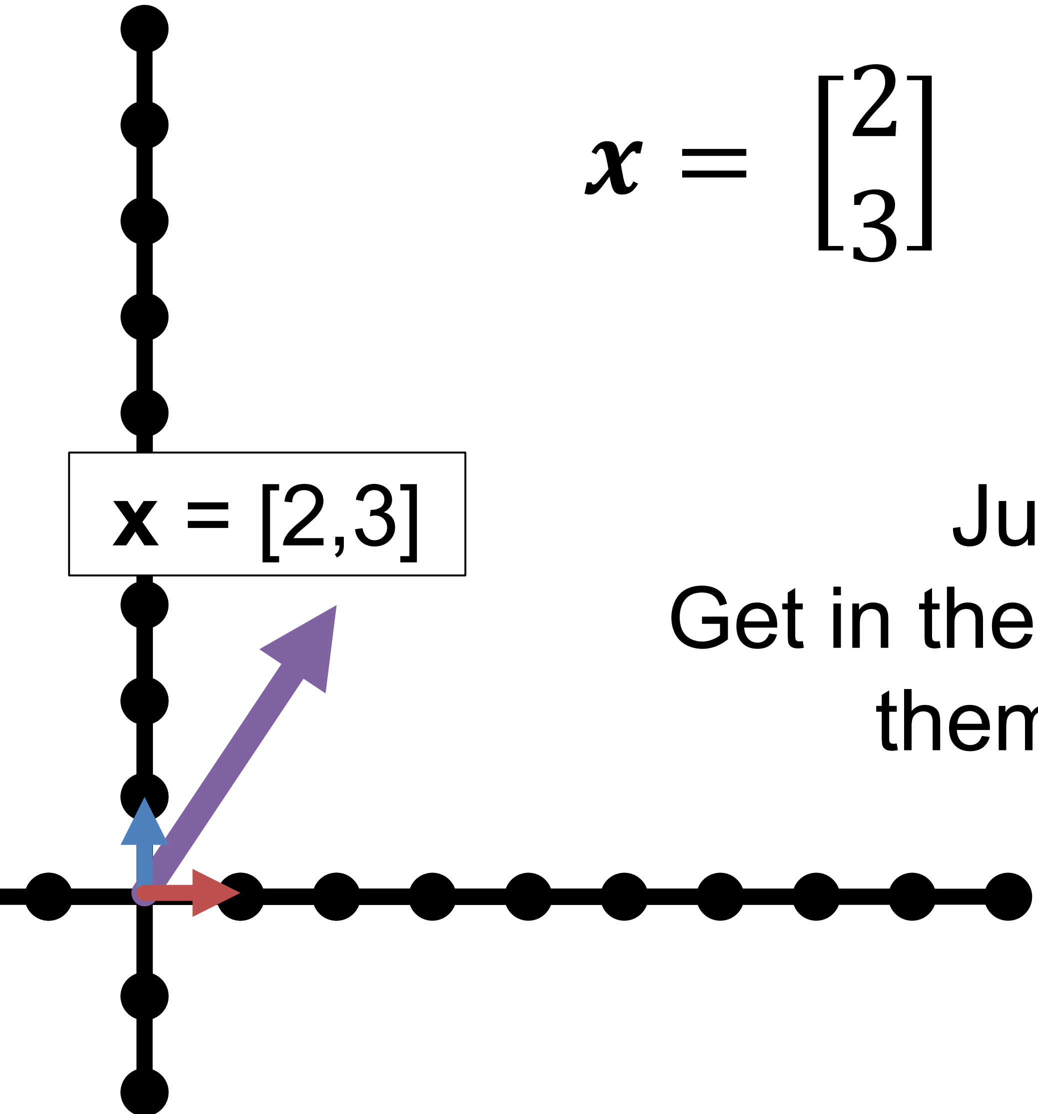
$$2 \times e_1 + 3 \times e_2$$

Can be arbitrary # of dimensions (typically denoted R^n)



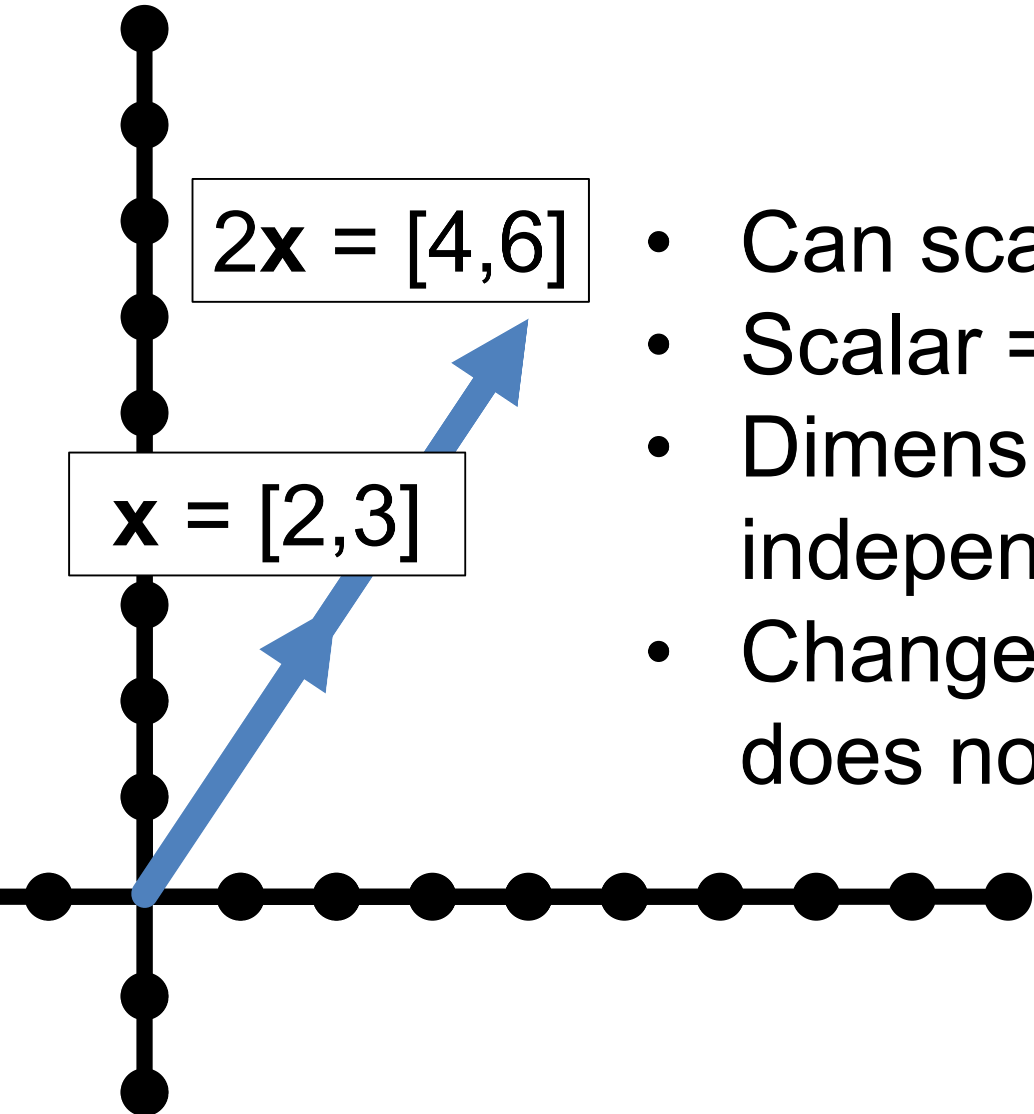
Vectors

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \boldsymbol{x}_1 = 2 \quad \boldsymbol{x}_2 = 3$$



Just an array!
Get in the habit of thinking of them as columns.

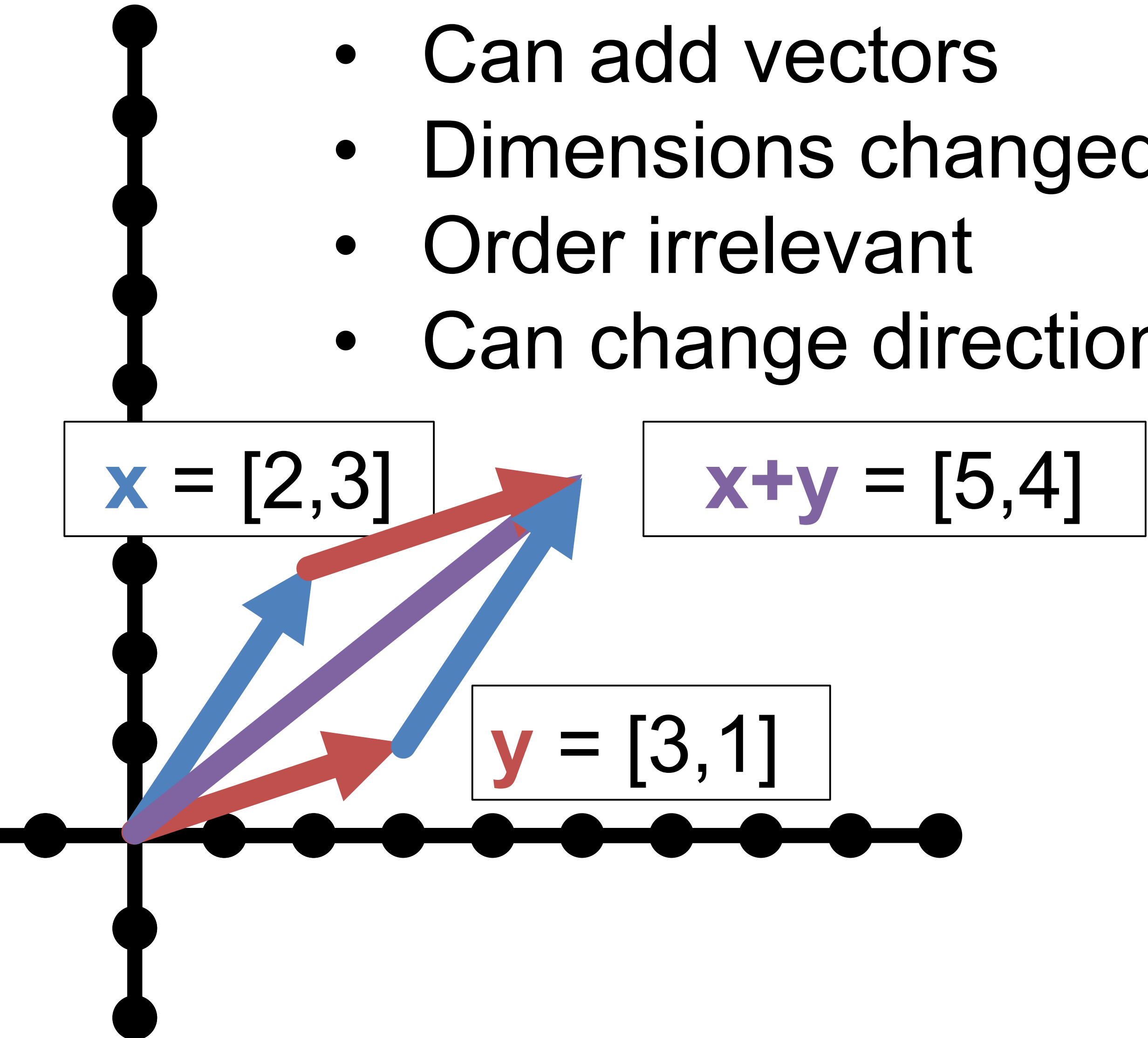
Scaling Vectors



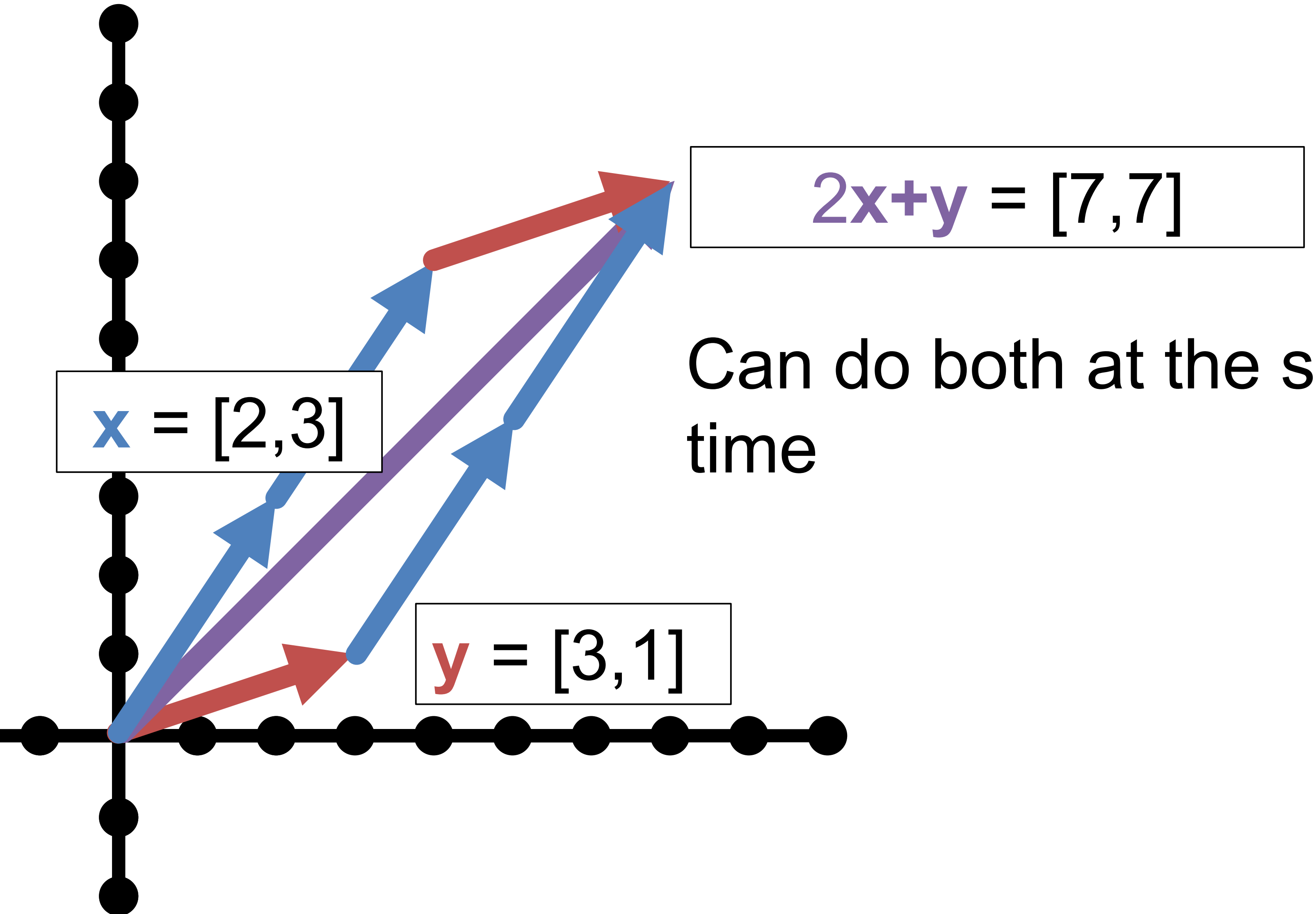
- Can scale vector by a *scalar*
- Scalar = single number
- Dimensions changed independently
- Changes *magnitude / length*, does not change *direction*.

Adding Vectors

- Can add vectors
- Dimensions changed independently
- Order irrelevant
- Can change direction and magnitude



Scaling and Adding



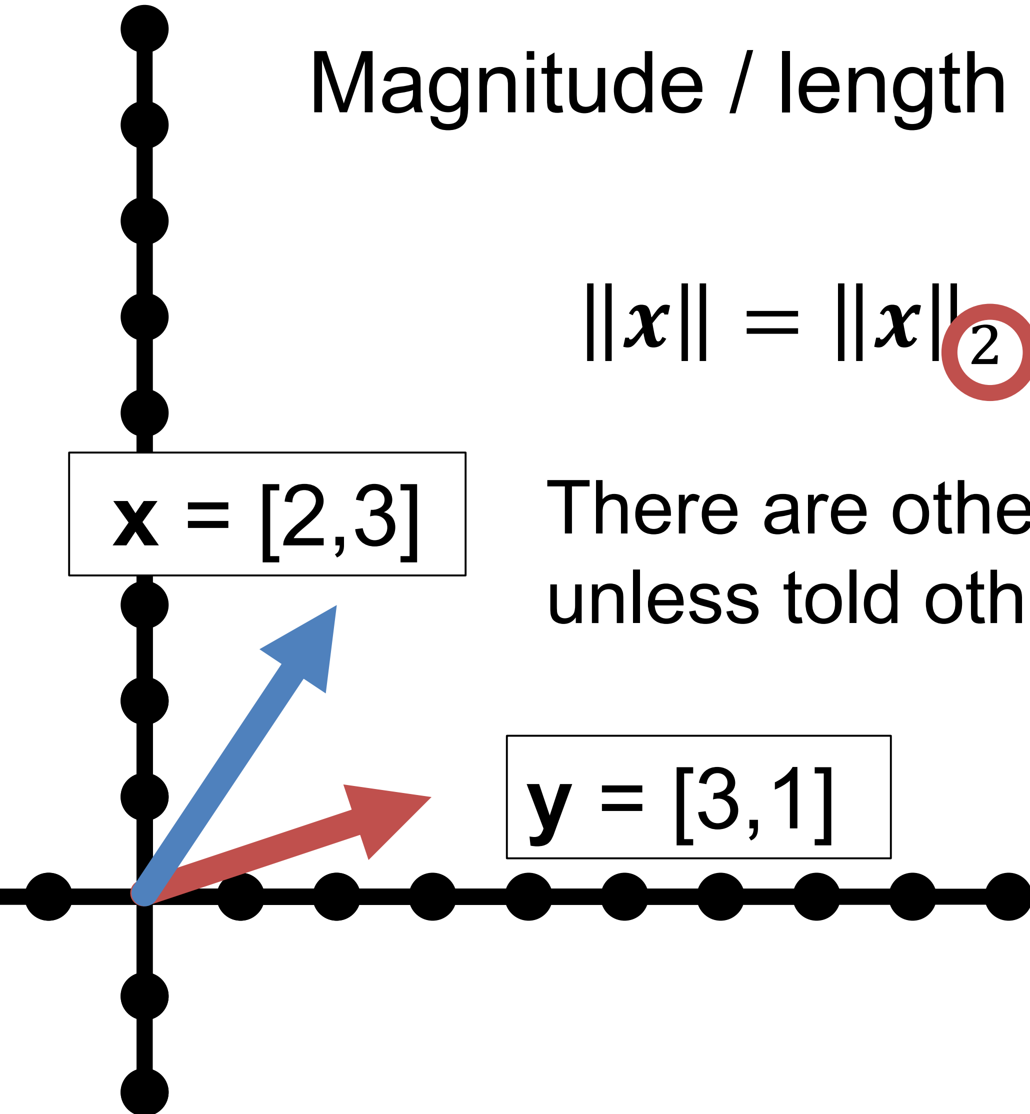
Can do both at the same time

Measuring Length

Magnitude / length / (L2) norm of vector

$$\|\boldsymbol{x}\| = \|\boldsymbol{x}\|_2 = \left(\sum_i^n x_i^2\right)^{1/2}$$

There are other norms; assume L2 unless told otherwise



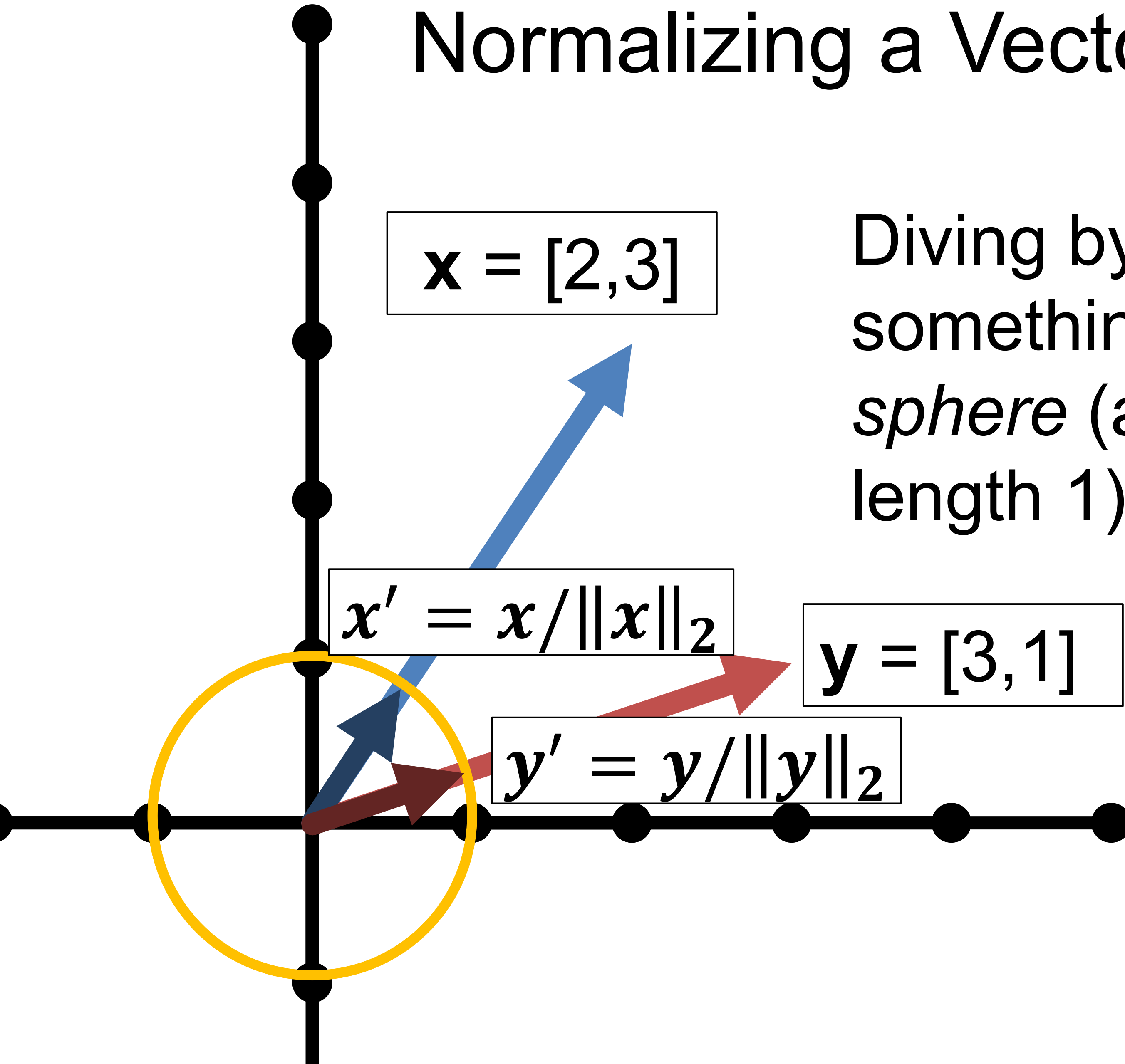
$$\|\boldsymbol{x}\|_2 = \sqrt{13}$$

$$\|\boldsymbol{y}\|_2 = \sqrt{10}$$

Why?

Normalizing a Vector

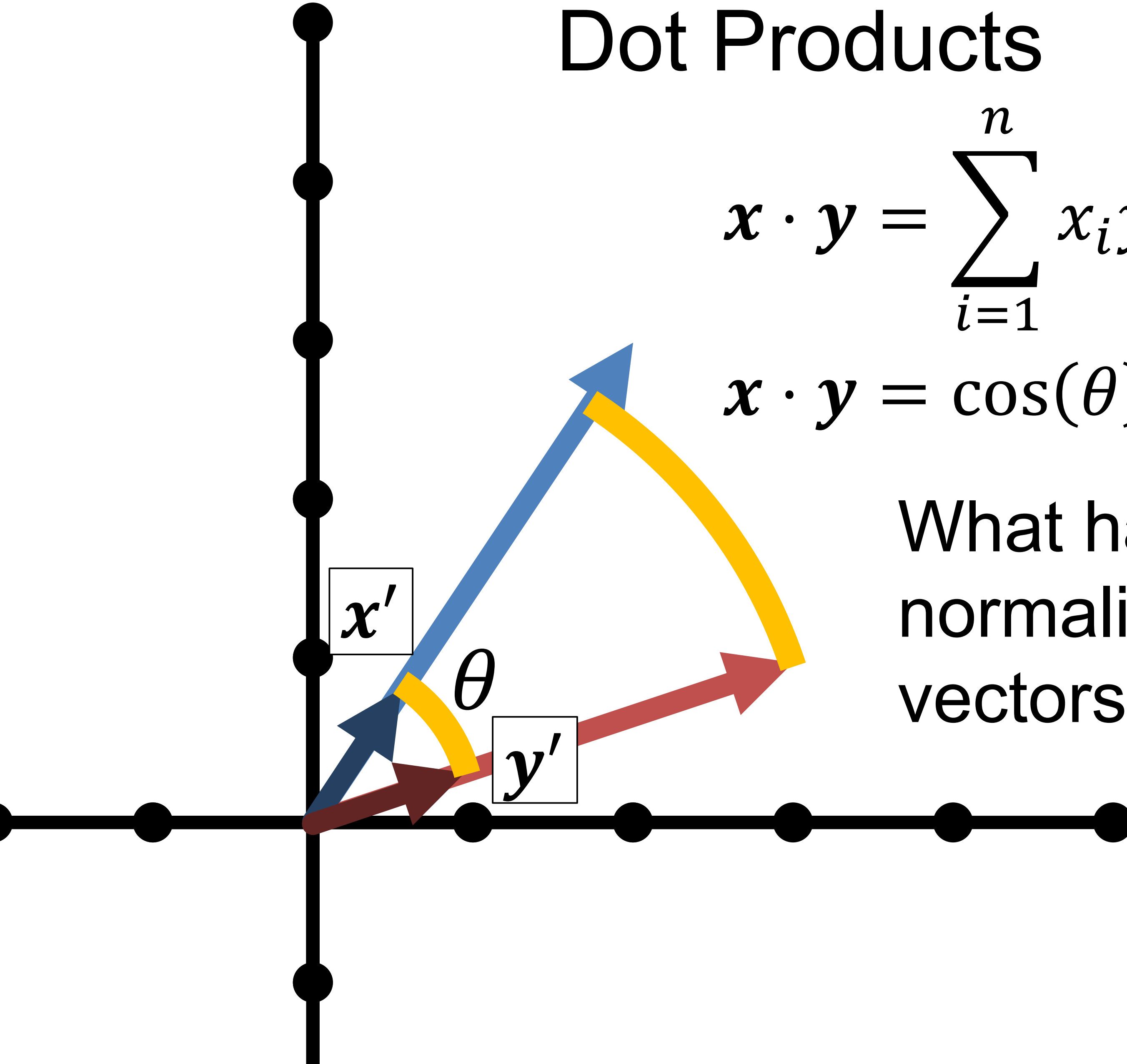
Diving by norm gives something on the *unit sphere* (all vectors with length 1)



Dot Products

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i = \boldsymbol{x}^T \boldsymbol{y}$$

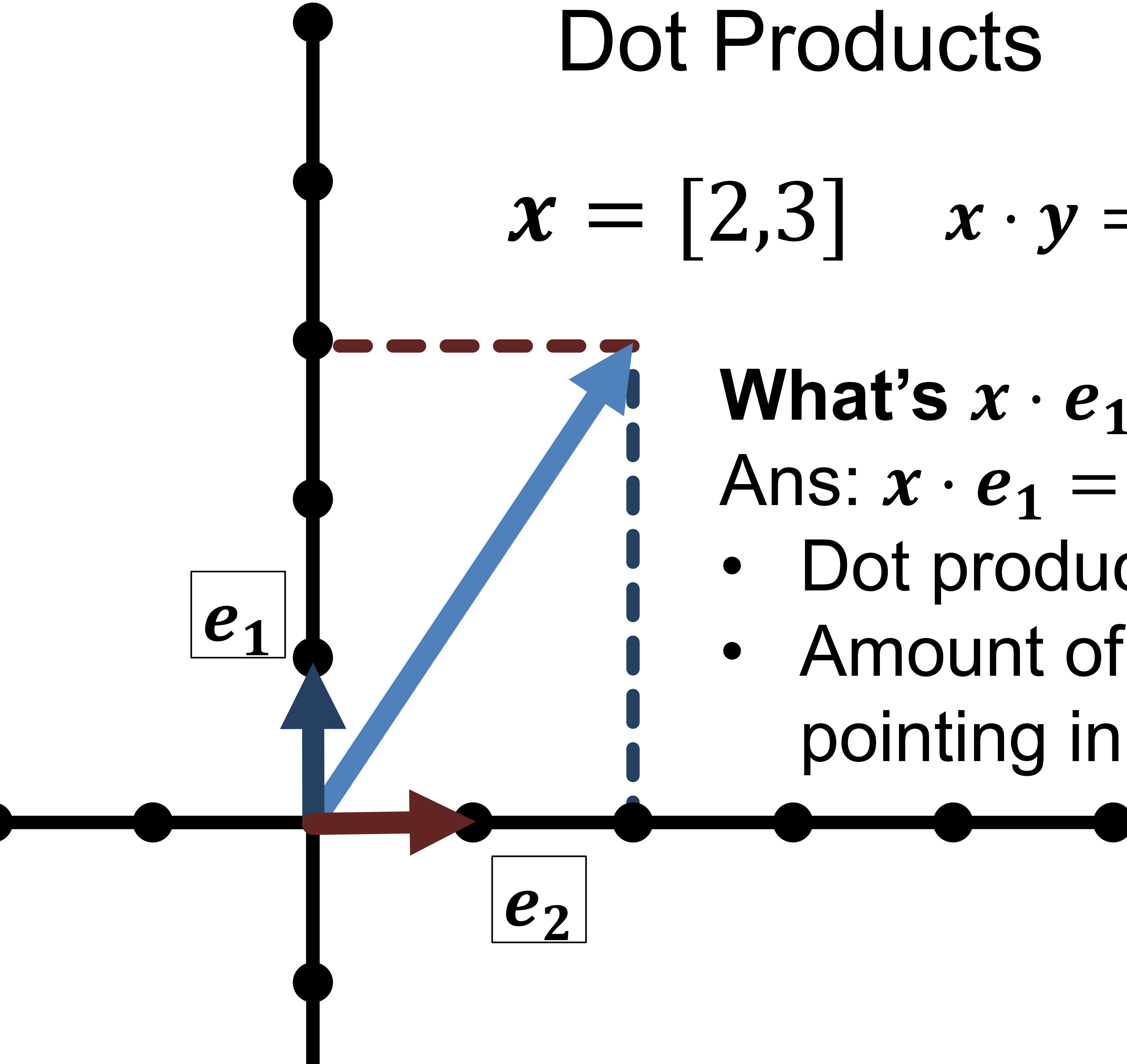
$$\boldsymbol{x} \cdot \boldsymbol{y} = \cos(\theta) \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$$



What happens with normalized / unit vectors?

Dot Products

$$\boldsymbol{x} = [2{,}3] \qquad \boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i}^{n} x_i y_i$$



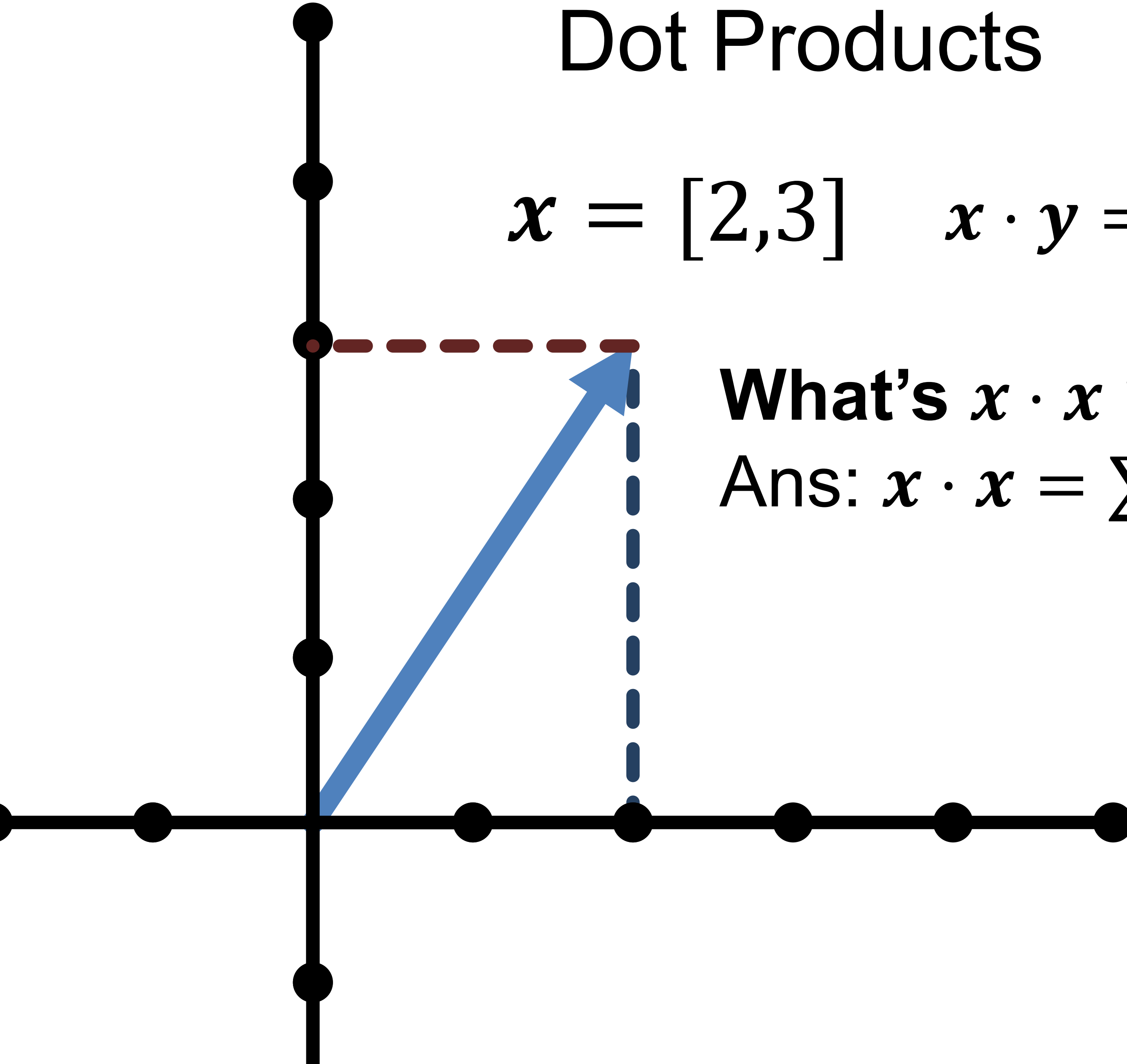
What's $\boldsymbol{x} \cdot \boldsymbol{e_1}$, $\boldsymbol{x} \cdot \boldsymbol{e_2}$?

Ans: $\boldsymbol{x} \cdot \boldsymbol{e_1} = 1$; $\boldsymbol{x} \cdot \boldsymbol{e_2} = 3$

- Dot product is projection
- Amount of **x** that's also pointing in direction of **y**

Dot Products

$$\boldsymbol{x} = [2,3] \quad \boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i}^{n} x_i y_i$$

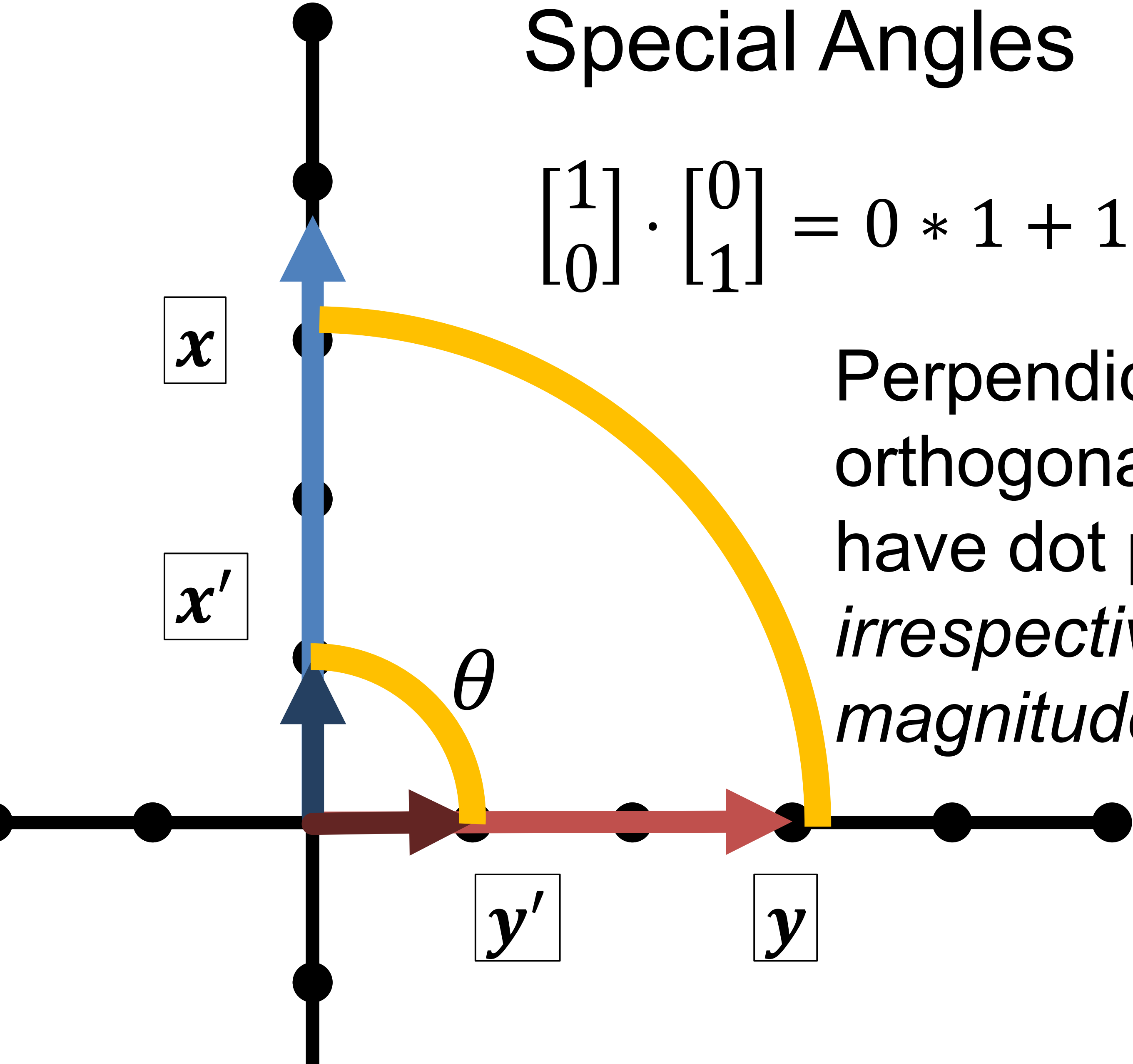


What's $\boldsymbol{x} \cdot \boldsymbol{x}$?

Ans: $\boldsymbol{x} \cdot \boldsymbol{x} = \sum x_i x_i = \|\boldsymbol{x}\|_2^2$

Special Angles

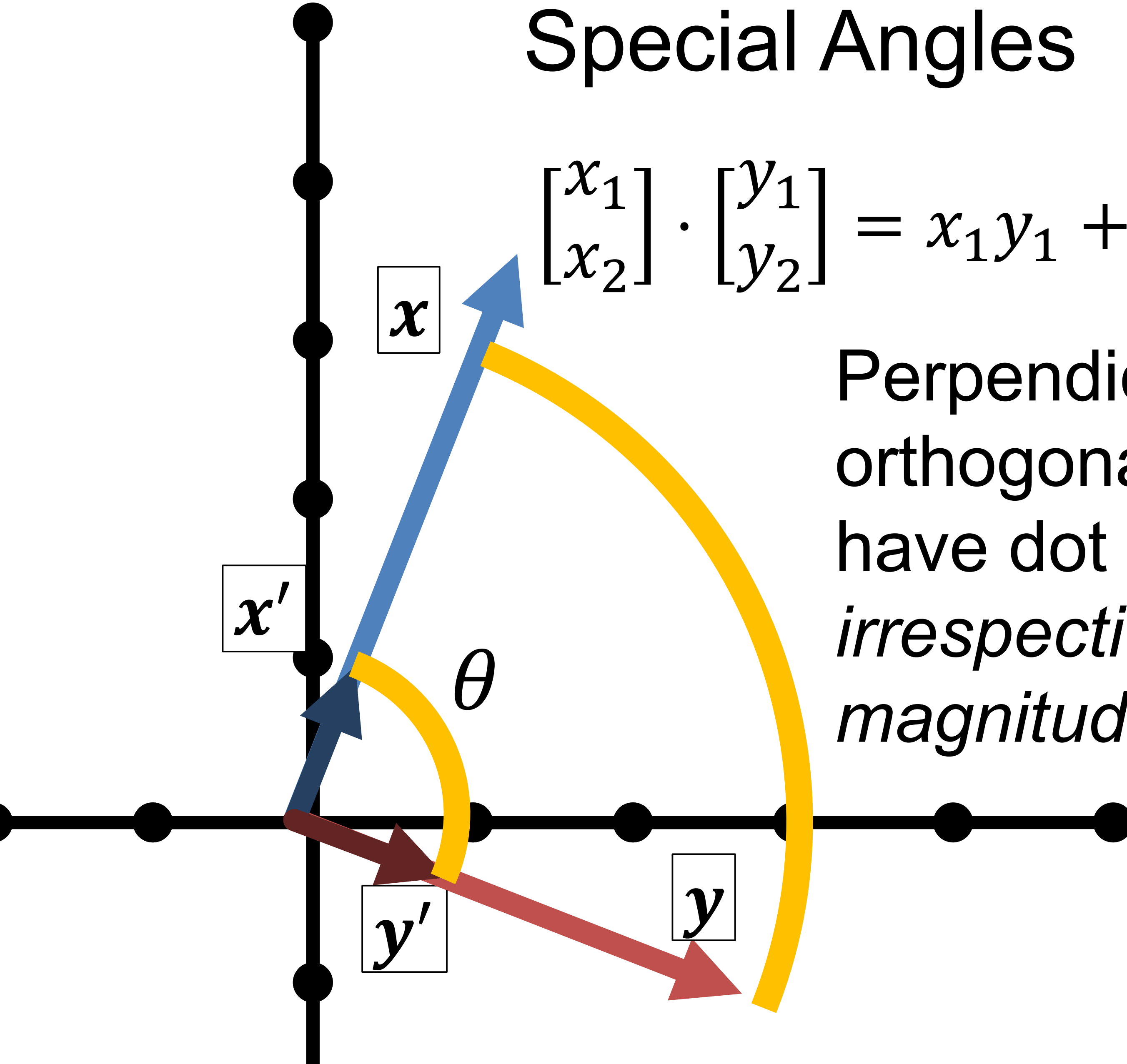
$$\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix} = 0 * 1 + 1 * 0 = 0$$



Perpendicular / orthogonal vectors have dot product 0 *irrespective of their magnitude*

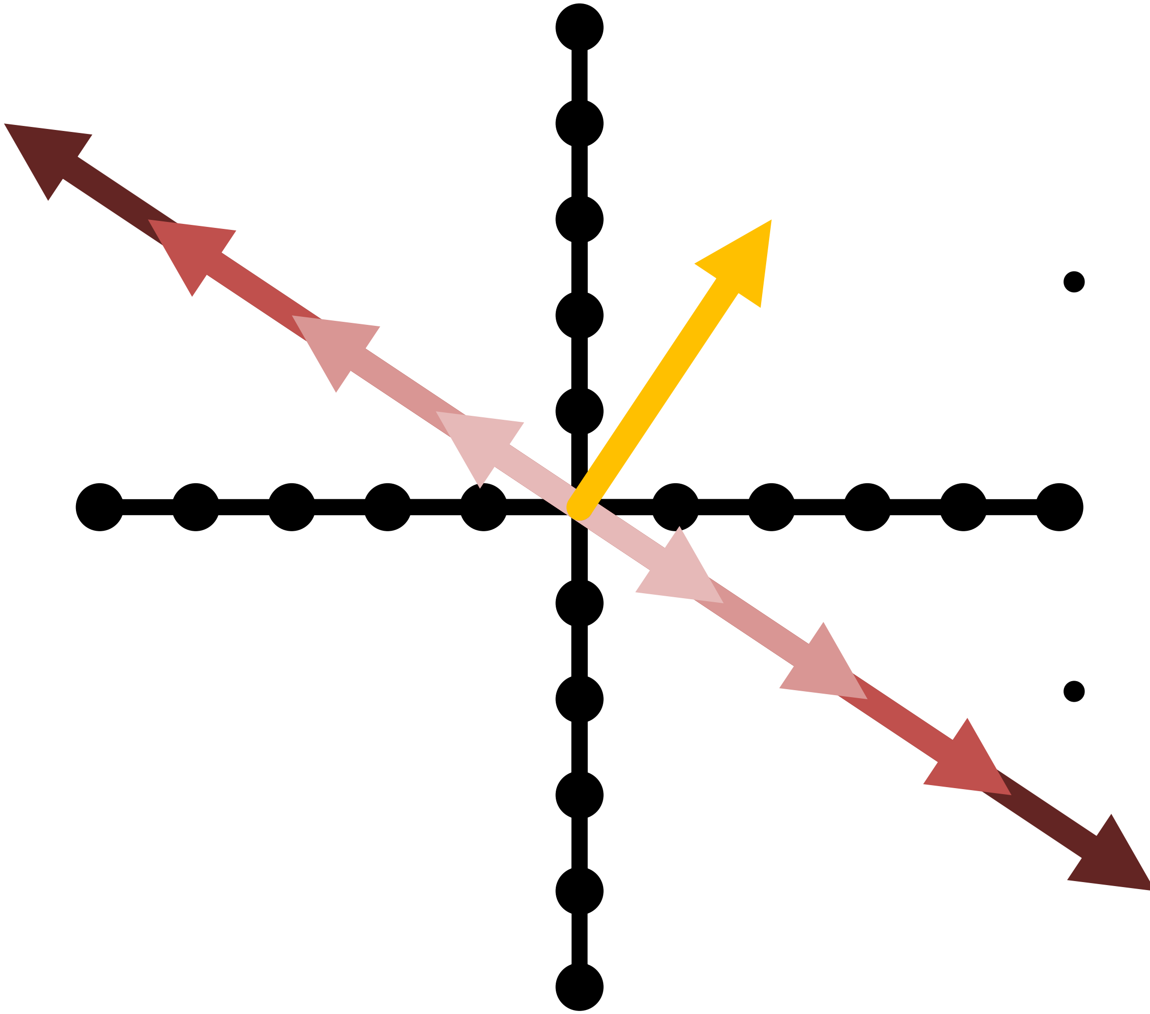
Special Angles

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + x_2 y_2 = 0$$



Perpendicular / orthogonal vectors have dot product 0 *irrespective of their magnitude*

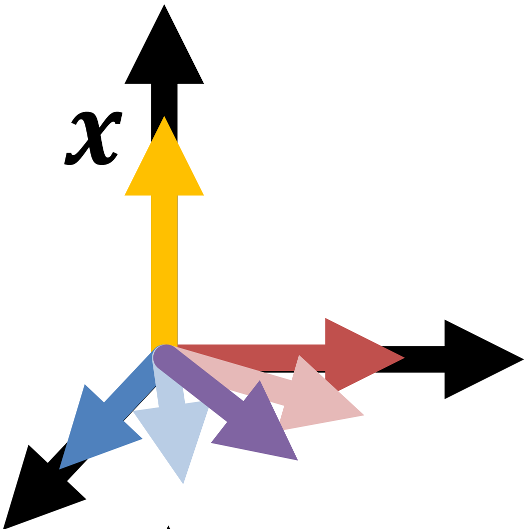
Orthogonal Vectors



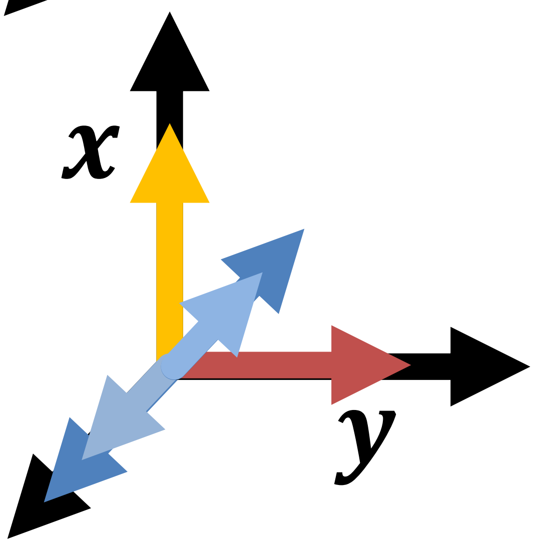
$\boldsymbol{x} = [2,3]$

- **Geometrically, what's the set of vectors that are orthogonal to x?**
- A line [3,-2]

Orthogonal Vectors

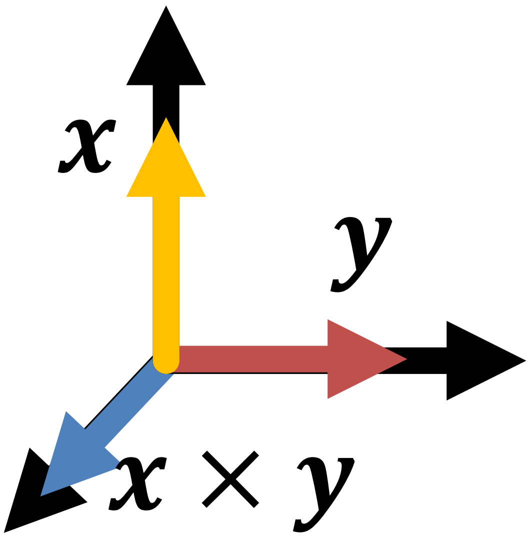


- **What's the set of vectors that are orthogonal to x = [5,0,0]?**
- A plane/2D space of vectors/any vector $[0, a, b]$



- **What's the set of vectors that are orthogonal to x <u>and</u> y = [0,5,0]?**
- A line/1D space of vectors/any vector $[0,0, b]$
- Ambiguity in *sign and magnitude*

Cross Product



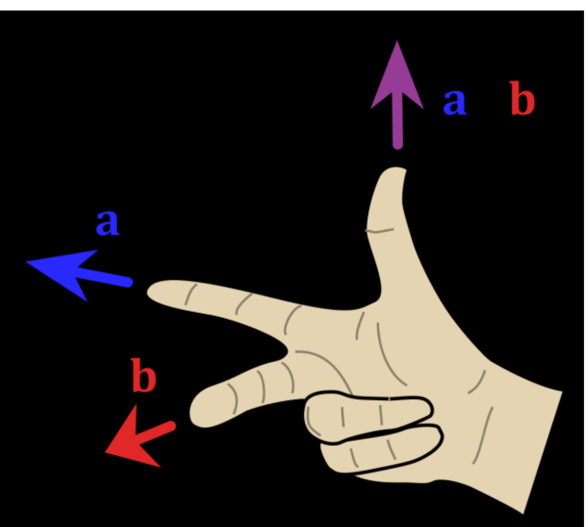


Image credit: Wikipedia.org

- Set $\{z: z \cdot x = 0, z \cdot y = 0\}$ has an ambiguity in sign and magnitude
- Cross product $x \times y$ is: (1) orthogonal to x, y (2) has sign given by right hand rule and (3) has magnitude given by area of parallelogram of **x** and **y**
- **Important**: if x and y are the same direction or either is **0**, then $x \times y = \mathbf{0}$.
- Only in 3D!

Operations You Should Know

- Scale (vector, scalar → vector)
- Add (vector, vector → vector)
- Magnitude (vector → scalar)
- Dot product (vector, vector → scalar)
  - Dot products are projection / angles
- Cross product (vector, vector → vector)
  - Vectors facing same direction have cross product **0**
- You can **never** mix vectors of different sizes

Matrices

Horizontally concatenate n, m-dim column vectors and you get a mxn matrix A (here 2x3)

$$\boldsymbol{A} = [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n] = \begin{bmatrix} v_{1_1} & v_{2_1} & v_{3_1} \\ v_{1_2} & v_{2_2} & v_{3_2} \end{bmatrix}$$

a (scalar) lowercase undecorated

a (vector) lowercase **bold or arrow**

A (matrix) uppercase **bold**

Matrices

Transpose: flip rows / columns

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T = \begin{bmatrix} a & b & c \end{bmatrix} \quad (3x1)^T = 1x3$$

Vertically concatenate m, n-dim row vectors and you get a mxn matrix A (here 2x3)

$$A = \begin{bmatrix} \boldsymbol{u}_1^T \\ \vdots \\ \boldsymbol{u}_n^T \end{bmatrix} = \begin{bmatrix} u_{1_1} & u_{1_2} & u_{1_3} \\ u_{2_1} & u_{2_2} & u_{2_3} \end{bmatrix}$$

Matrix-Vector Product

$$\boldsymbol{y}_{2x1} = \boldsymbol{A}_{2x3}\boldsymbol{x}_{3x1}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\boldsymbol{y} = x_1\boldsymbol{v_1} + x_2\boldsymbol{v_2} + x_3\boldsymbol{v_3}$$

Linear combination of columns of ***A***

Matrix-Vector Product

$$\boldsymbol{y}_{2x1} = \boldsymbol{A}_{2x3}\boldsymbol{x}_{3x1}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \quad \boldsymbol{u_1} \quad \\ \quad \boldsymbol{u_2} \quad \end{bmatrix} \boldsymbol{x}$$

$$y_1 = \boldsymbol{u_1^T x} \qquad y_2 = \boldsymbol{u_2^T x}$$

*Dot product between rows of **A** and **x***

Matrix Multiplication

Generally: $\boldsymbol{A}_{mn}$ *and* $\boldsymbol{B}_{np}$ *yield product* $(\boldsymbol{AB})_{mp}$

$$\boldsymbol{AB} = \begin{bmatrix} - & \boldsymbol{a}_1^T & - \\ & \vdots & \\ - & \boldsymbol{a}_m^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_p \\ | & & | \end{bmatrix}$$

Yes – in **A**, I'm referring to the rows, and in **B**, I'm referring to the columns

Matrix Multiplication

Generally: $\boldsymbol{A}_{mn}$ *and* $\boldsymbol{B}_{np}$ *yield product* $(\boldsymbol{AB})_{mp}$

$$\begin{bmatrix} | & & | \\ \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_p \\ | & & | \end{bmatrix}$$

$$\boldsymbol{AB} = \begin{bmatrix} - & \boldsymbol{a}_1^T & - \\ & \vdots & \\ - & \boldsymbol{a}_m^T & - \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1^T\boldsymbol{b}_1 & \cdots & \boldsymbol{a}_1^T\boldsymbol{b}_p \\ \vdots & \ddots & \vdots \\ \boldsymbol{a}_m^T\boldsymbol{b}_1 & \cdots & \boldsymbol{a}_m^T\boldsymbol{b}_p \end{bmatrix}$$

$$\boldsymbol{AB}_{ij} = \boldsymbol{a}_i^T\boldsymbol{b}_j$$

Matrix Multiplication

- Dimensions must match
- Dimensions must match
- Dimensions must match
- (Yes, it's associative): ABx = (A)(Bx) = (AB)x
- (*No it's not commutative*): ABx ≠ (BA)x ≠ (BxA)

Operations They Don't Teach

You Probably Saw Matrix Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

What is this? FYI: e is a scalar

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + e = \qquad \begin{bmatrix} a+e & b+e \\ c+e & d+e \end{bmatrix}$$

Broadcasting

If you want to be pedantic and proper, you expand e by multiplying a matrix of 1s (denoted **1**)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + e = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \mathbf{1}_{2x2} e$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & e \\ e & e \end{bmatrix}$$

Many smart matrix libraries do this automatically.
This is the source of many bugs.

Broadcasting Example

Given: a nx2 matrix **P** and a 2D column vector **v**,
Want: nx2 difference matrix **D**

$$\boldsymbol{P} = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} x_1 - a & y_1 - b \\ \vdots & \vdots \\ x_n - a & y_n - b \end{bmatrix}$$

$$\boldsymbol{P} - \boldsymbol{v}^T = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{matrix} [a & b] \\ \vdots & \\ [a & b] \end{matrix}$$

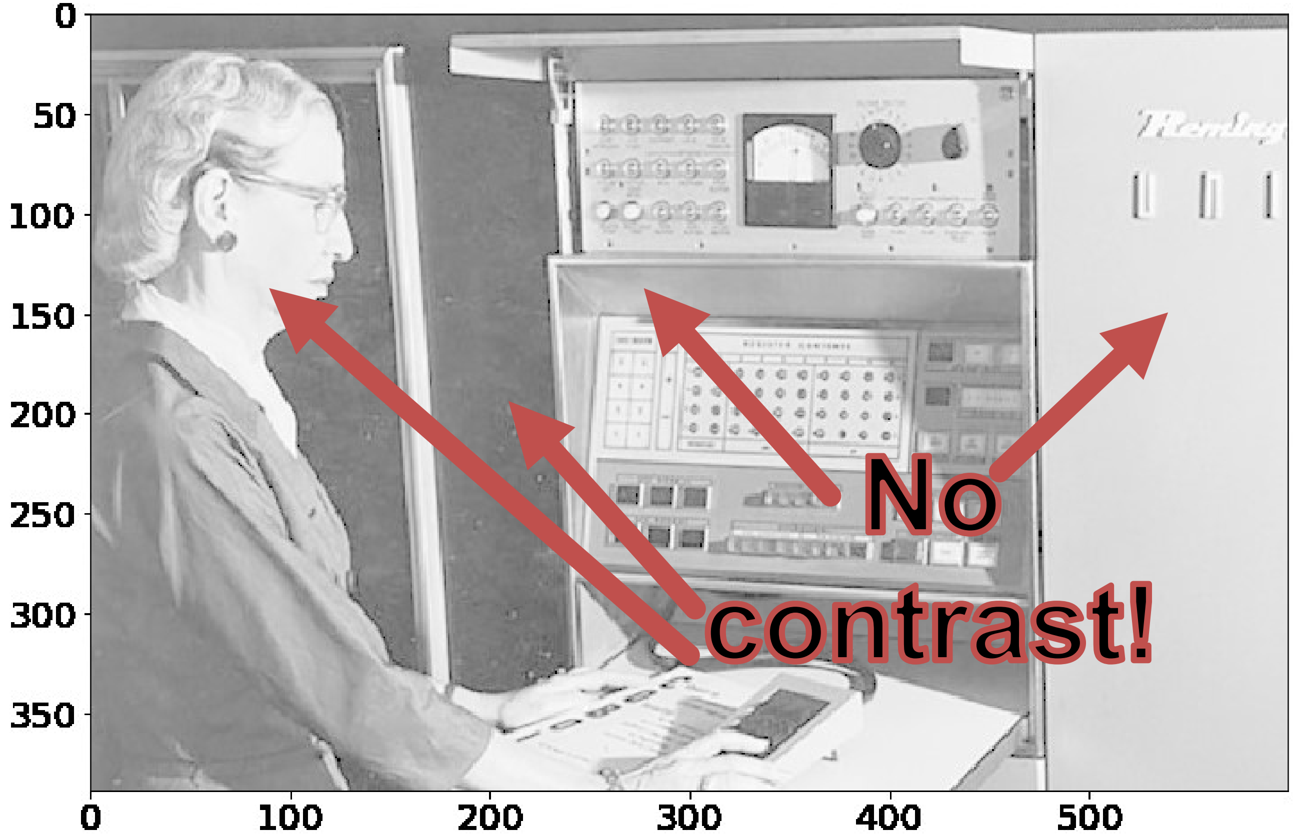
Blue stuff is assumed / broadcast

Two Uses for Matrices

1. Storing things in a rectangular array (images, maps)
   - *Typical operations*: element-wise operations, convolution (which we'll cover next)
   - *Atypical operations*: almost anything you learned in a math linear algebra class
2. A linear operator that maps vectors to another space (**Ax**)
   - *Typical/Atypical:* reverse of above

Images as Matrices

Suppose someone hands you this matrix.
What's wrong with it?

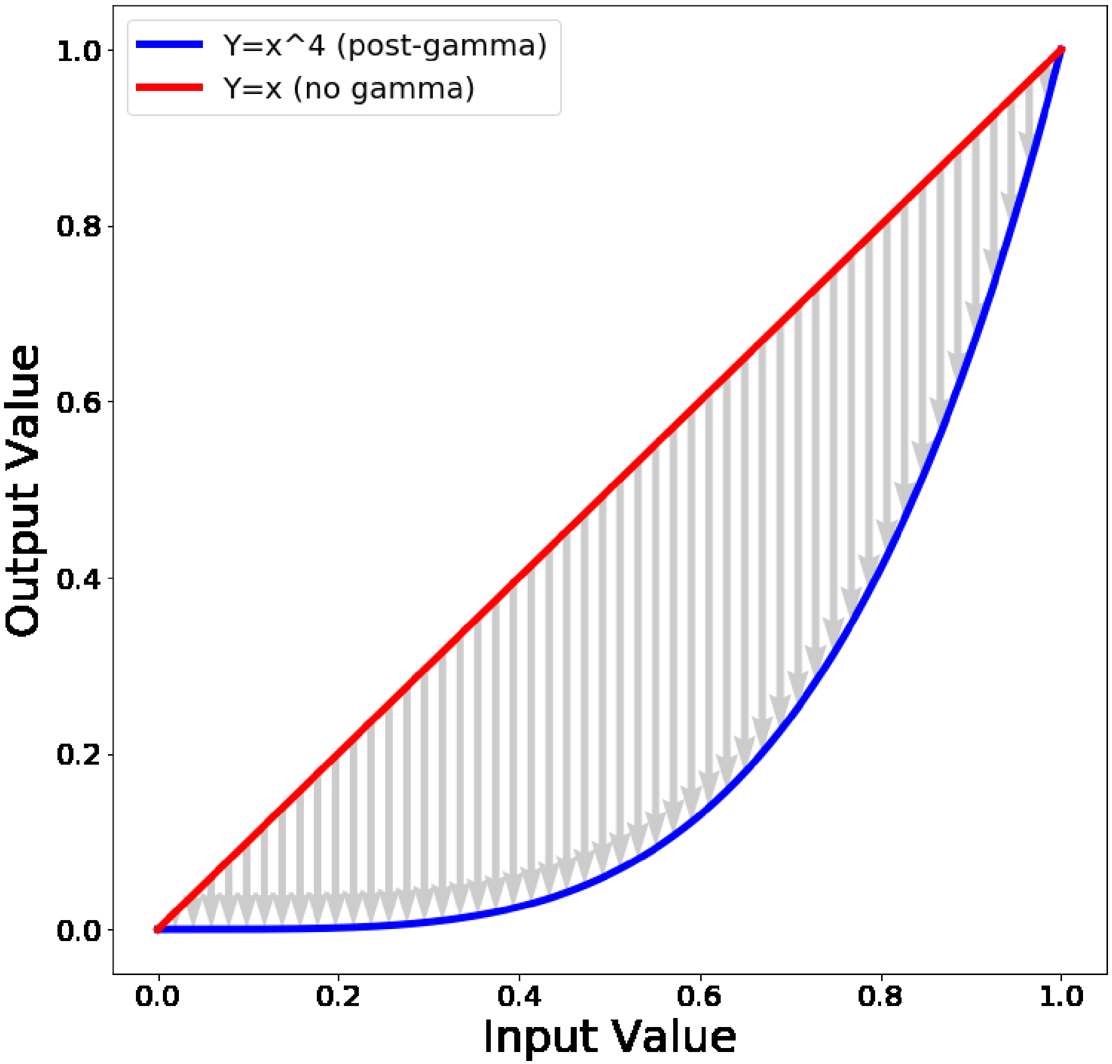


Contrast – Gamma curve

Typical way to change the contrast is to apply a nonlinear correction

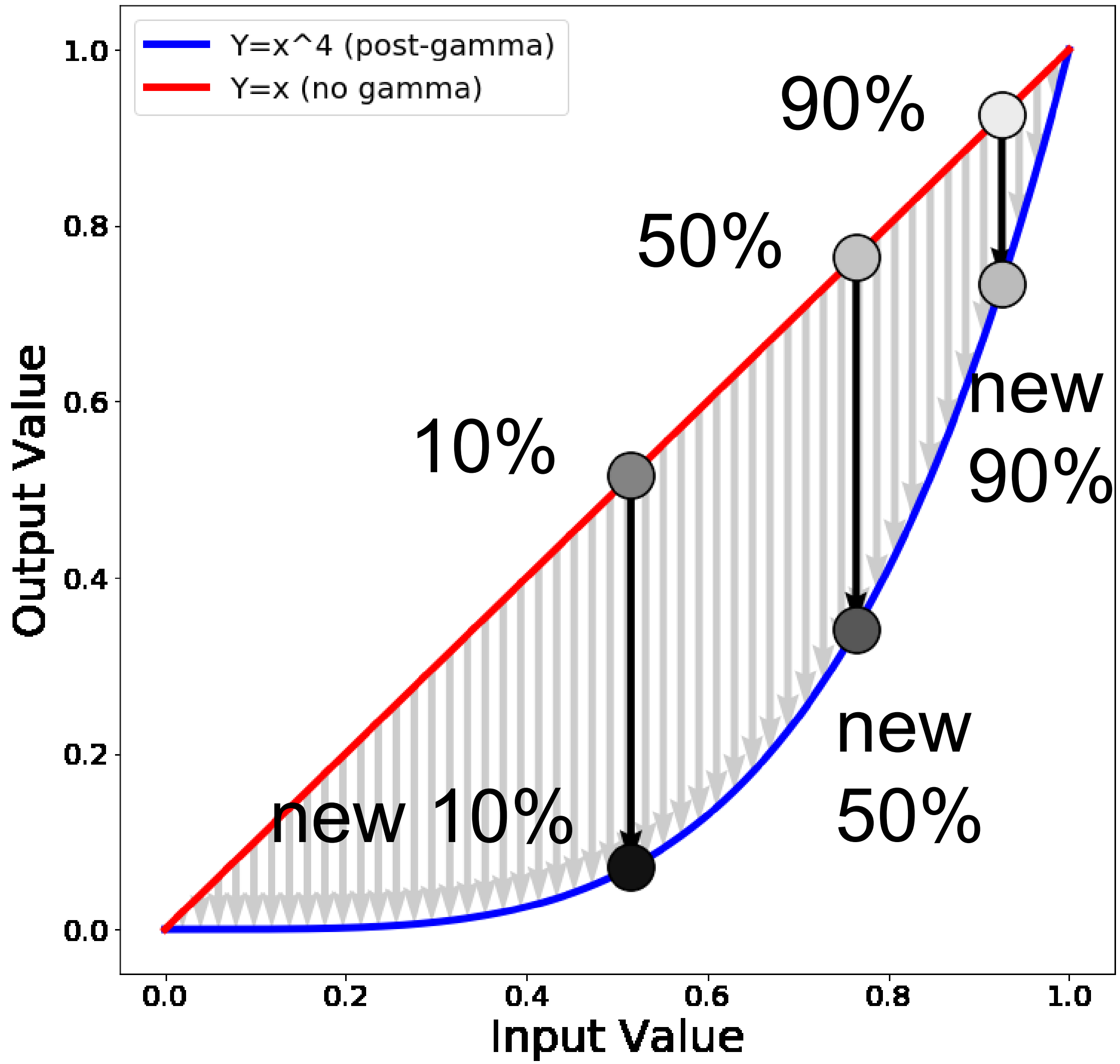
$$\text{pixelvalue}^{\gamma}$$

The quantity γ controls how much contrast gets added



Contrast – Gamma curve

Now the darkest regions (10th pctile) are **much** darker than the moderately dark regions (50th pctile).



Implementation

Python+Numpy (right way):

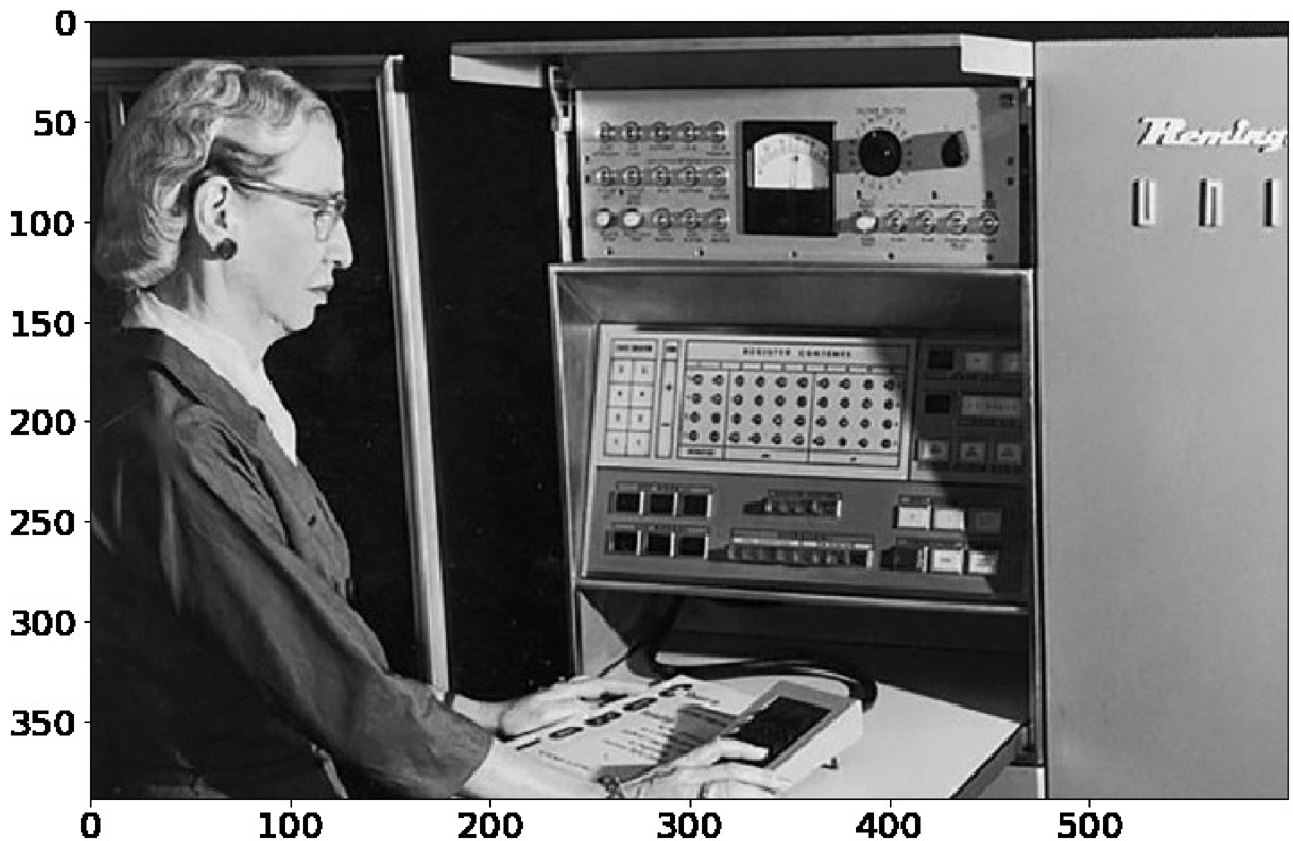
```
imNew = im**0.25
```

Python+Numpy (slow way – **why?**):

```
imNew = np.zeros(im.shape)
for y in range(im.shape[0]):
  for x in range(im.shape[1]):
    imNew[y,x] = im[y,x]**expFactor
```

Results

Phew! Much Better.



Element-wise Operations

Element-wise power – beware notation

$$(\boldsymbol{A}^p)_{ij} = A_{ij}^p$$

“Hadamard Product” / Element-wise multiplication

$$(\boldsymbol{A} \odot \boldsymbol{B})_{ij} = \boldsymbol{A}_{ij} * \boldsymbol{B}_{ij}$$

Element-wise division

$$(\boldsymbol{A}/\boldsymbol{B})_{ij} = \frac{A_{ij}}{B_{ij}}$$

Sums Across Axes

Suppose have Nx2 matrix **A**

$$\boldsymbol{A} = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

ND col. vec.

$$\Sigma(\boldsymbol{A}, 1) = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

2D row vec

$$\Sigma(\boldsymbol{A}, 0) = \left[\sum_{i=1}^{n} x_i \quad , \sum_{i=1}^{n} y_i \right]$$

Note – libraries distinguish between N-D column vector and Nx1 matrix.

Vectorizing Example

- Suppose I represent each image as a 128-dimensional vector
- I want to compute all the pairwise distances between {$\mathbf{x}_1, \ldots, \mathbf{x}_N$} and {$\mathbf{y}_1, \ldots, \mathbf{y}_M$} so I can find, for every $\mathbf{x}_i$ the nearest $\mathbf{y}_j$
- Identity: $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x}^T\boldsymbol{y}$
- Or: $\|\boldsymbol{x} - \boldsymbol{y}\| = (\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x}^T\boldsymbol{y})^{1/2}$

Vectorizing Example

$$\boldsymbol{X} = \begin{bmatrix} - & \boldsymbol{x}_1 & - \\ & \vdots & \\ - & \boldsymbol{x}_N & - \end{bmatrix} \quad \boldsymbol{Y} = \begin{bmatrix} - & \boldsymbol{y}_1 & - \\ & \vdots & \\ - & \boldsymbol{y}_M & - \end{bmatrix} \quad \boldsymbol{Y}^T = \begin{bmatrix} | & & | \\ \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_M \\ | & & | \end{bmatrix}$$

Compute a Nx1 vector of norms (can also do Mx1)

$$\boldsymbol{\Sigma}(\boldsymbol{X}^2, \boldsymbol{1}) = \begin{bmatrix} \|\boldsymbol{x}_1\|^2 \\ \vdots \\ \|\boldsymbol{x}_N\|^2 \end{bmatrix}$$

Compute a NxM matrix of dot products

$$\left(\boldsymbol{X}\boldsymbol{Y}^T\right)_{ij} = \boldsymbol{x}_i^T \boldsymbol{y}_j$$

Vectorizing Example

$$\mathbf{D} = \left(\Sigma(\boldsymbol{X}^{\mathbf{2}}, 1) + \Sigma(\boldsymbol{Y}^{\mathbf{2}}, 1)^{\boldsymbol{T}} - 2\boldsymbol{X}\boldsymbol{Y}^{\boldsymbol{T}}\right)^{1/2}$$

$$\begin{bmatrix} \|\boldsymbol{x}_{\mathbf{1}}\|^{\mathbf{2}} \\ \vdots \\ \|\boldsymbol{x}_N\|^{\mathbf{2}} \end{bmatrix} + \begin{bmatrix} \|\boldsymbol{y}_1\|^{\mathbf{2}} & \cdots & \|\boldsymbol{y}_M\|^{\mathbf{2}} \end{bmatrix}$$

$$\begin{bmatrix} \|\boldsymbol{x}_{\mathbf{1}}\|^2 + \|\boldsymbol{y}_{\mathbf{1}}\|^2 & \cdots & \|\boldsymbol{x}_{\mathbf{1}}\|^2 + \|\boldsymbol{y}_{\boldsymbol{M}}\|^2 \\ \vdots & \ddots & \vdots \\ \|\boldsymbol{x}_N\|^2 + \|\boldsymbol{y}_{\mathbf{1}}\|^2 & \cdots & \|\boldsymbol{x}_N\|^2 + \|\boldsymbol{y}_{\boldsymbol{M}}\|^2 \end{bmatrix}$$

Why?

$$(\Sigma(X^2, 1) + \Sigma(Y^2, 1)^T)_{ij} = \|x_i\|^2 + \|y_j\|^2$$

Vectorizing Example

$$\mathbf{D} = \left(\Sigma(\boldsymbol{X}^2, 1) + \Sigma(\boldsymbol{Y}^2, 1)^T - 2\boldsymbol{X}\boldsymbol{Y}^T\right)^{1/2}$$

$$\mathbf{D}_{ij} = \|\boldsymbol{x}_i\|^2 + \left\|\boldsymbol{y}_j\right\|^2 + 2\boldsymbol{x}^T\boldsymbol{y}$$

Numpy code:

```
XNorm = np.sum(X**2,axis=1,keepdims=True)
YNorm = np.sum(Y**2,axis=1,keepdims=True)
D = (XNorm+YNorm.T-2*np.dot(X,Y.T))**0.5
```

*May have to make sure this is at least 0 (sometimes roundoff issues happen)

Does it Make a Difference?

Computing pairwise distances between 300 and 400 128-dimensional vectors

1. for x in X, for y in Y, using native python: 9s
2. for x in X, for y in Y, using numpy to compute distance: 0.8s
3. vectorized: 0.0045s (~2000x faster than 1, 175x faster than 2)

Expressing things in primitives that are optimized is usually faster

Linear Independence

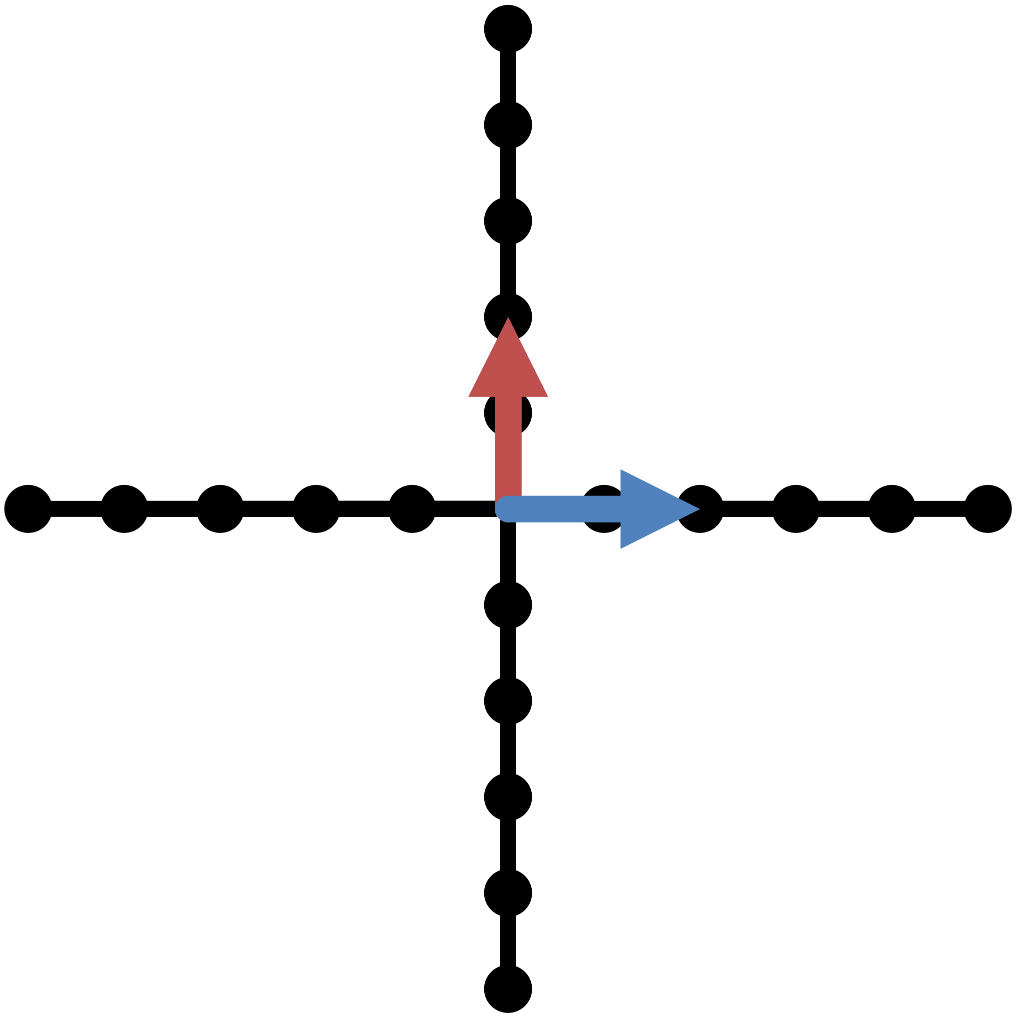
A set of vectors is linearly independent if you can't write one as a linear combination of the others.

Suppose: $\boldsymbol{a} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ $\boldsymbol{b} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$ $\boldsymbol{c} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 2\boldsymbol{a} \qquad \boldsymbol{y} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b}$$

- **Is the set {a,b,c} linearly independent?**
- **Is the set {a,b,x} linearly independent?**
- **Max # of independent 3D vectors?**

Span

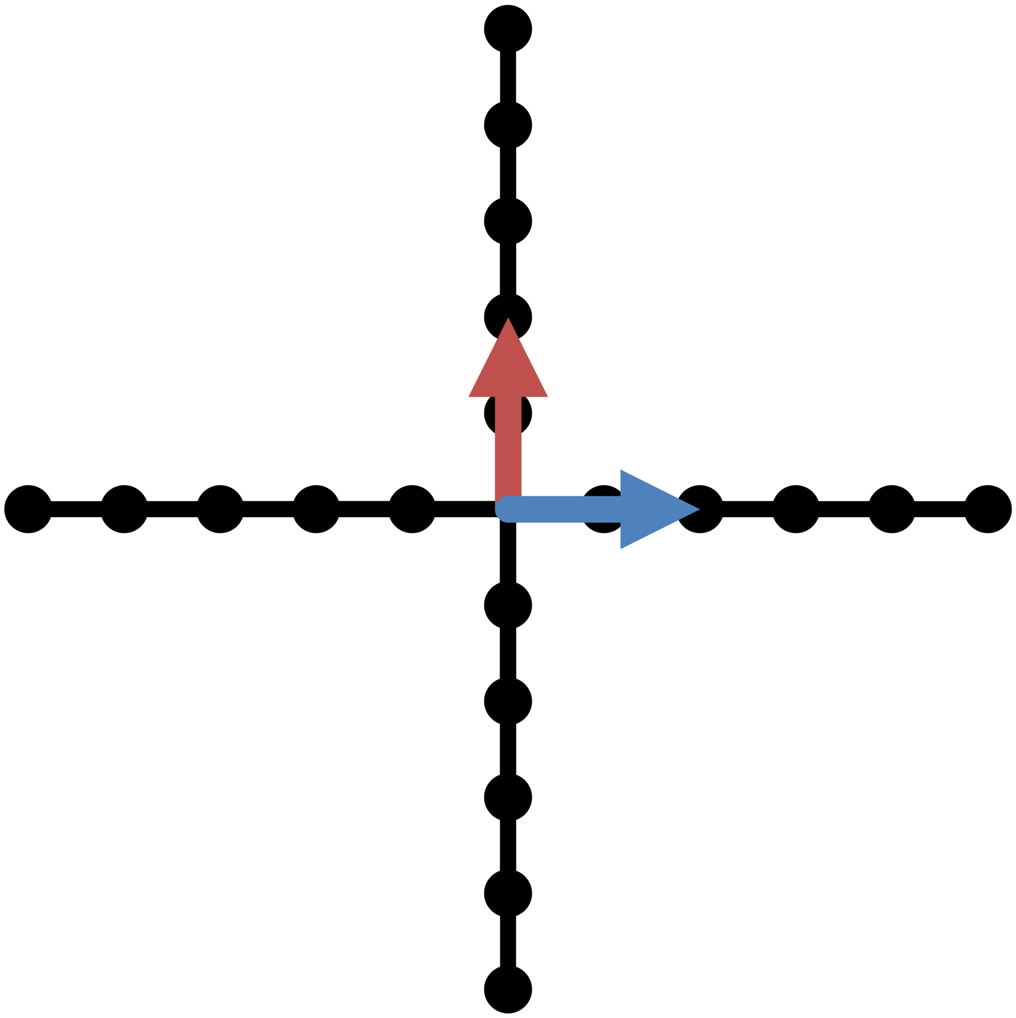


Span: all linear combinations of a set of vectors

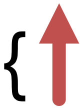
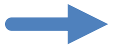
Span({ ↑ }) =
Span({[0,1]}) = ?
All vertical lines through origin =
$\{\lambda[0,1]: \lambda \in R\}$

Is blue in {red}'s span?

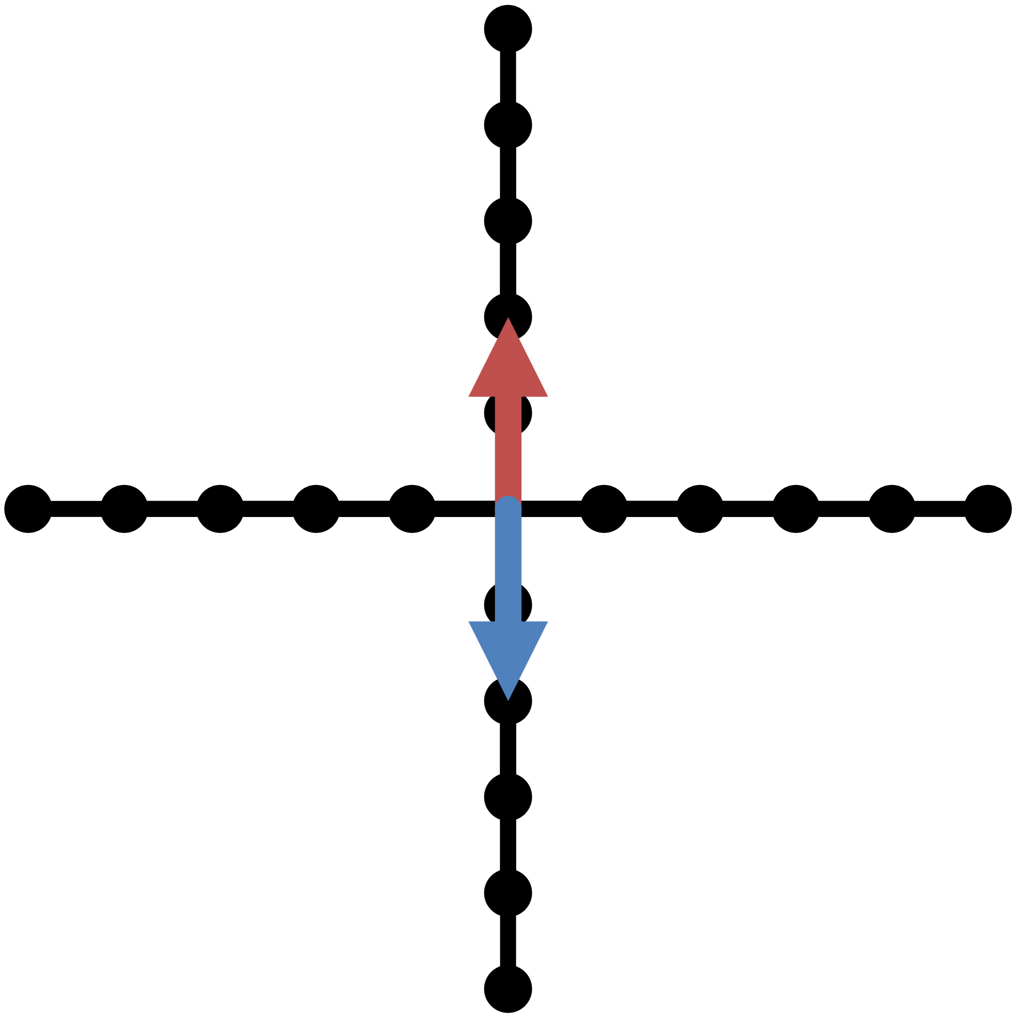
Span



Span: all linear combinations of a set of vectors

Span({  ,  }) = **?**

Span



Span: all linear combinations of a set of vectors

Span({↑ , ↓ }) = **?**

Matrix-Vector Product

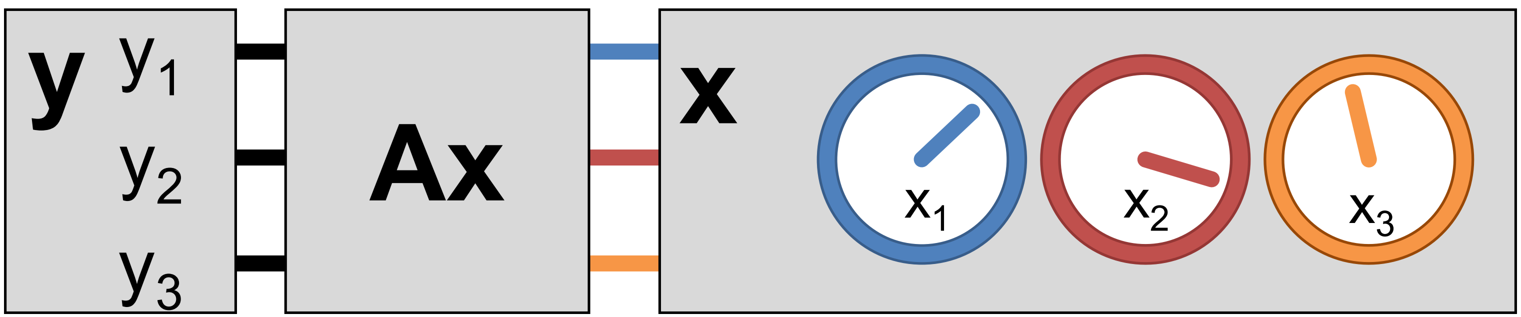
$$\boldsymbol{Ax} = \begin{bmatrix} | & & | \\ \boldsymbol{c_1} & \cdots & \boldsymbol{c_n} \\ | & & | \end{bmatrix} \boldsymbol{x}$$

Right-multiplying **A** by **x** mixes columns of **A** according to entries of **x**

- The output space of f(**x**) = **Ax** is constrained to be the *span* of the columns of **A**.
- Can’t output things you can’t construct out of your columns

An Intuition

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} | & | & | \\ \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_n \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



x – knobs on machine (e.g., fuel, brakes)
y – state of the world (e.g., where you are)
A – machine (e.g., your car)

Linear Independence

Suppose the columns of 3x3 matrix **A** are *not* linearly independent (c_1, αc_1, c_2 for instance)

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} | & | & | \\ \boldsymbol{c_1} & \alpha\boldsymbol{c_1} & \boldsymbol{c_2} \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

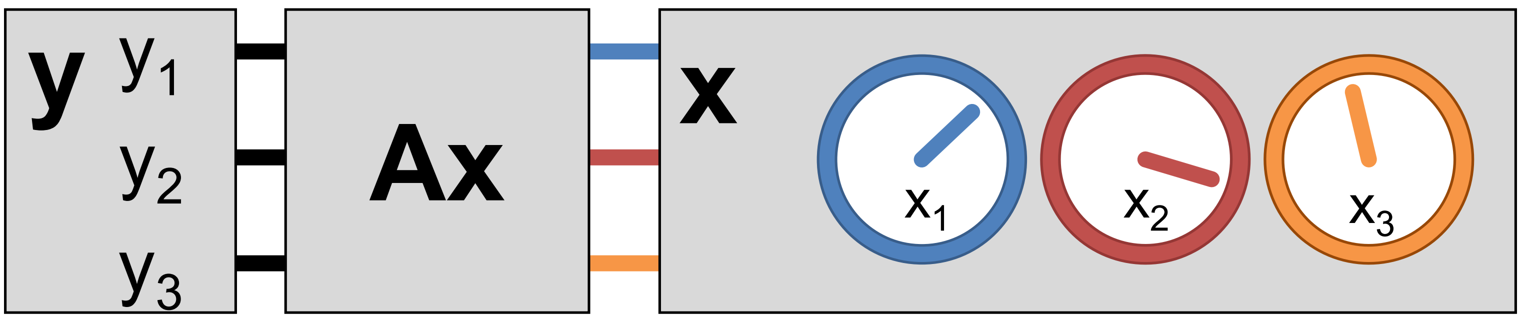
$$\boldsymbol{y} = x_1\boldsymbol{c_1} + \alpha x_2\boldsymbol{c_1} + x_3\boldsymbol{c_2}$$

$$\boldsymbol{y} = (x_1 + \alpha x_2)\boldsymbol{c_1} + x_3\boldsymbol{c_2}$$

Linear Independence Intuition

Knobs of **x** are redundant. Even if **y** has 3 outputs, you can only control it in two directions

$$\boldsymbol{y} = (x_1 + \alpha x_2)\boldsymbol{c_1} + x_3 \boldsymbol{c_2}$$



Linear Independence

Recall: $\boldsymbol{Ax} = (x_1 + \alpha x_2)\boldsymbol{c_1} + x_3\boldsymbol{c_2}$

$$\boldsymbol{y} = \boldsymbol{A}\begin{bmatrix} x_1 + \beta \\ x_2 - \beta/\alpha \\ x_3 \end{bmatrix} = \left(x_1 + \cancel{\beta} + \alpha x_2 - \cancel{\alpha\frac{\beta}{\alpha}}\right)c_1 + x_3 c_2$$

- Can write **y** an infinite number of ways by adding β to $\mathbf{x_1}$ and subtracting $\frac{\beta}{\alpha}$ from $\mathbf{x_2}$
- Or, given a vector **y** there's not a unique vector **x** s.t. **y** =**Ax**
- Not all **y** have a corresponding **x** s.t. **y=Ax**

Linear Independence

$$\boldsymbol{Ax} = (x_1 + \alpha x_2)\boldsymbol{c_1} + x_3\boldsymbol{c_2}$$

$$\boldsymbol{y} = \boldsymbol{A}\begin{bmatrix} \beta \\ -\beta/\alpha \\ 0 \end{bmatrix} = \left(\beta - \alpha\frac{\beta}{\alpha}\right)\boldsymbol{c_1} + 0\boldsymbol{c_2}$$

- What else can we cancel out?
- An infinite number of non-zero vectors **x** can map to a zero-vector **y**
- Called the right null-space of A.

Rank

- Rank of a nxn matrix **A** – number of linearly independent columns (**or rows**) of A / the dimension of the span of the columns
- Matrices with *full rank* (n x n, rank n) behave nicely: can be inverted, span the full output space, are one-to-one.
- Matrices with *full rank* are machines where every knob is useful and every output state can be made by the machine

Inverses

- Given $y = Ax$, **y** is a linear combination of columns of **A** proportional to **x**. If A is full-rank, we should be able to invert this mapping.
- Given some **y** (output) and **A**, what **x** (inputs) produced it?
- $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$
- Note: if you don't need to compute it, **never ever compute it.** Solving for **x** is much faster and stable than obtaining $\mathbf{A}^{-1}$.

Symmetric Matrices

- Symmetric: $\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{A}$ or $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji}$
- Have **lots** of special properties

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Any matrix of the form $\boldsymbol{A} = \boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}$ is symmetric.

Quick check:

$$\boldsymbol{A}^{\boldsymbol{T}} = \left(\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}\right)^{\boldsymbol{T}}$$

$$\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{X}^{\boldsymbol{T}}\left(\boldsymbol{X}^{\boldsymbol{T}}\right)^{\boldsymbol{T}}$$

$$\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}$$

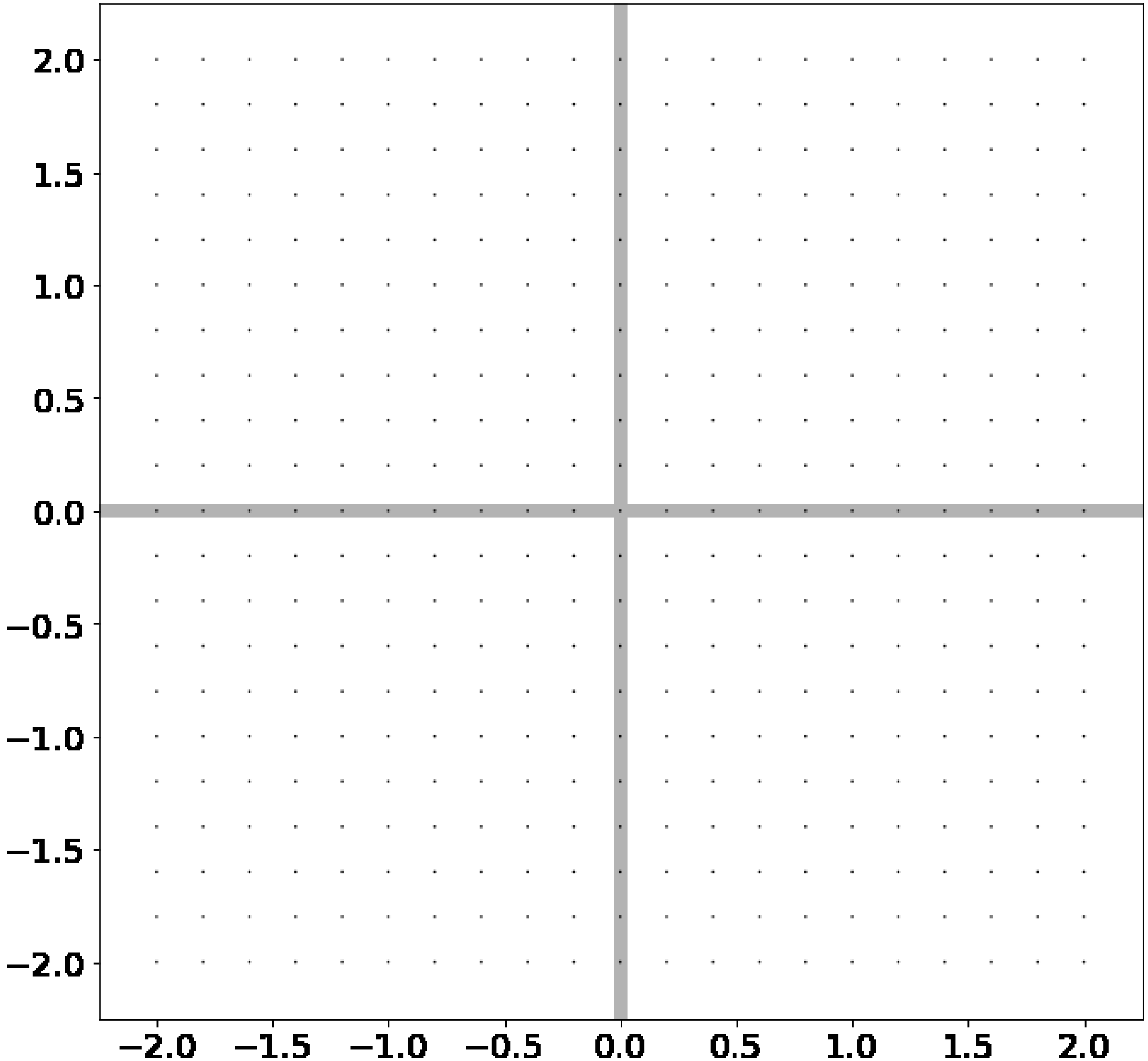
Special Matrices – Rotations

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

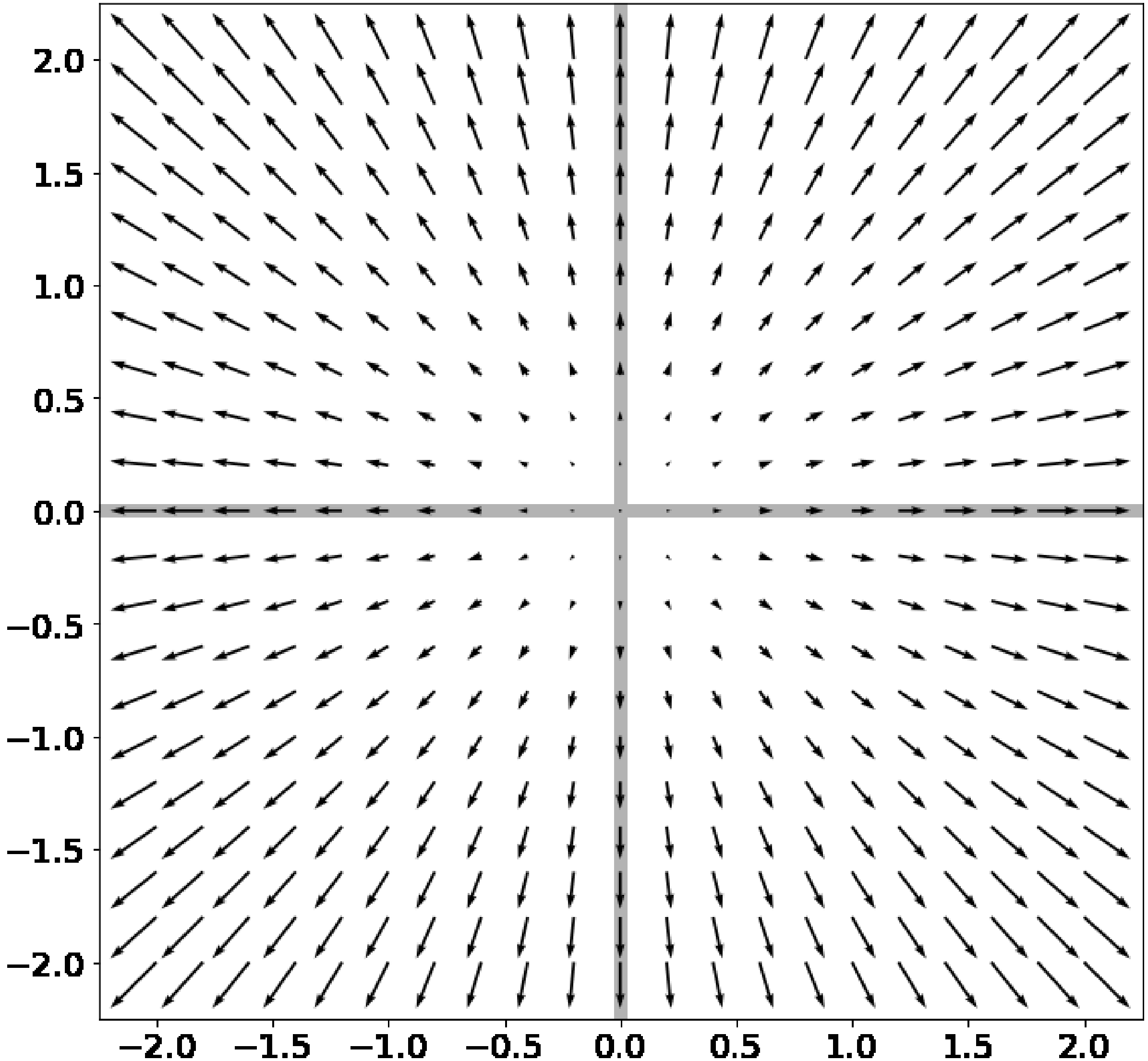
- Rotation matrices $\boldsymbol{R}$ rotate vectors and ***do not change vector L2 norms*** ($\|\boldsymbol{Rx}\|_2 = \|\boldsymbol{x}\|_2$)
- Every row/column is unit norm
- Every row is linearly independent
- Transpose is inverse $\boldsymbol{RR^T} = \boldsymbol{R^T R} = \boldsymbol{I}$
- Determinant is 1 (otherwise it's also a coordinate flip/reflection), eigenvalues are 1

Eigensystems

- An eigenvector $\boldsymbol{v_i}$ and eigenvalue λ_i of a matrix $\boldsymbol{A}$ satisfy $\boldsymbol{A v_i} = \lambda_i \boldsymbol{v_i}$ ($\boldsymbol{A v_i}$ is scaled by λ_i)
- Vectors and values are always paired and typically you assume $\|\boldsymbol{v_i}\|^2 = 1$
- Biggest eigenvalue of A gives bounds on how much $f(\boldsymbol{x}) = \boldsymbol{A x}$ stretches a vector $\mathbf{x}$.
- Hints of what people really mean:
  - "Largest eigenvector" = vector w/ largest value
  - Spectral just means there's eigenvectors



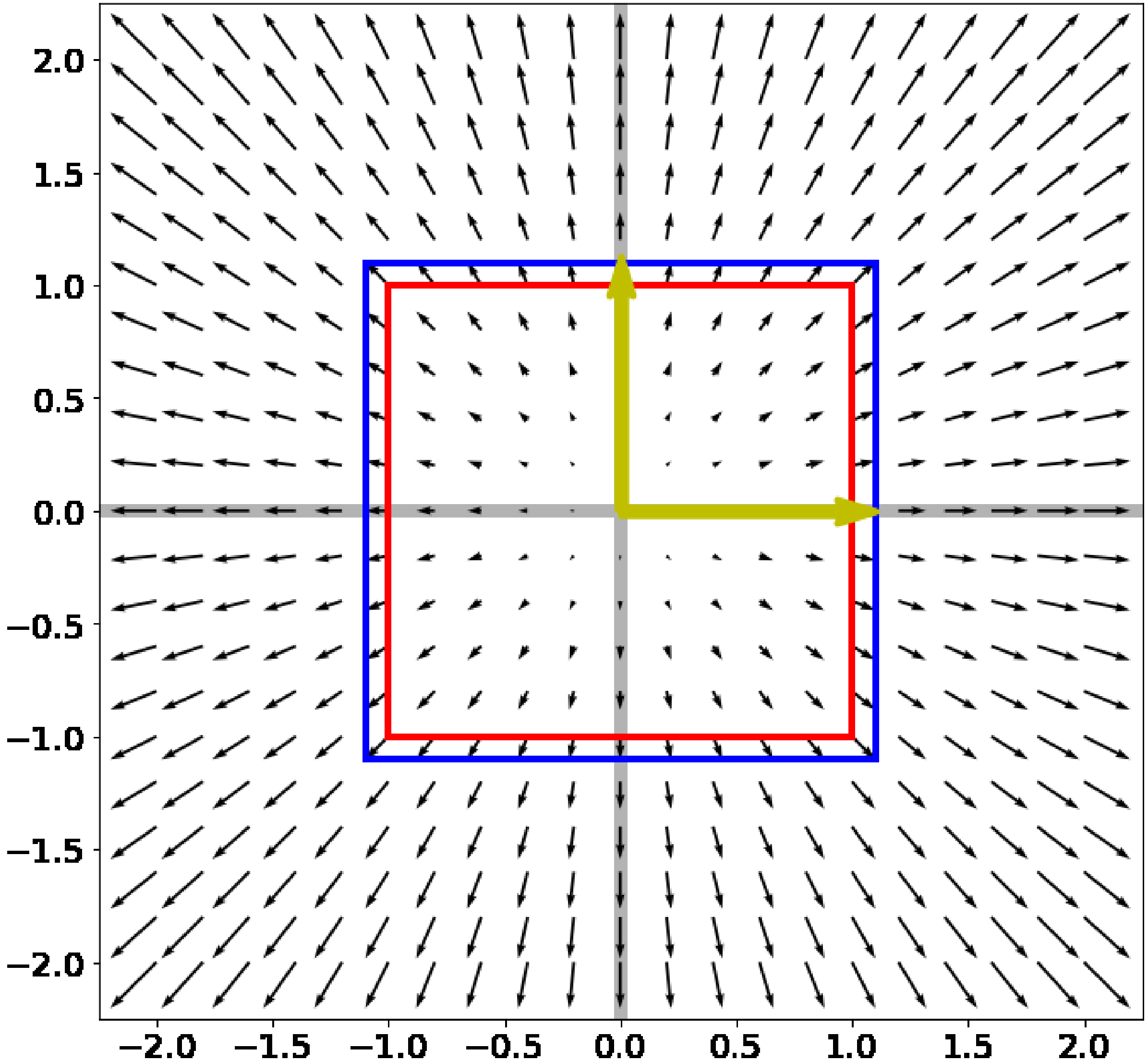
Suppose I have points in a grid



Now I apply f(**x**) = **Ax** to these points

Pointy-end: **Ax** . Non-Pointy-End: **x**

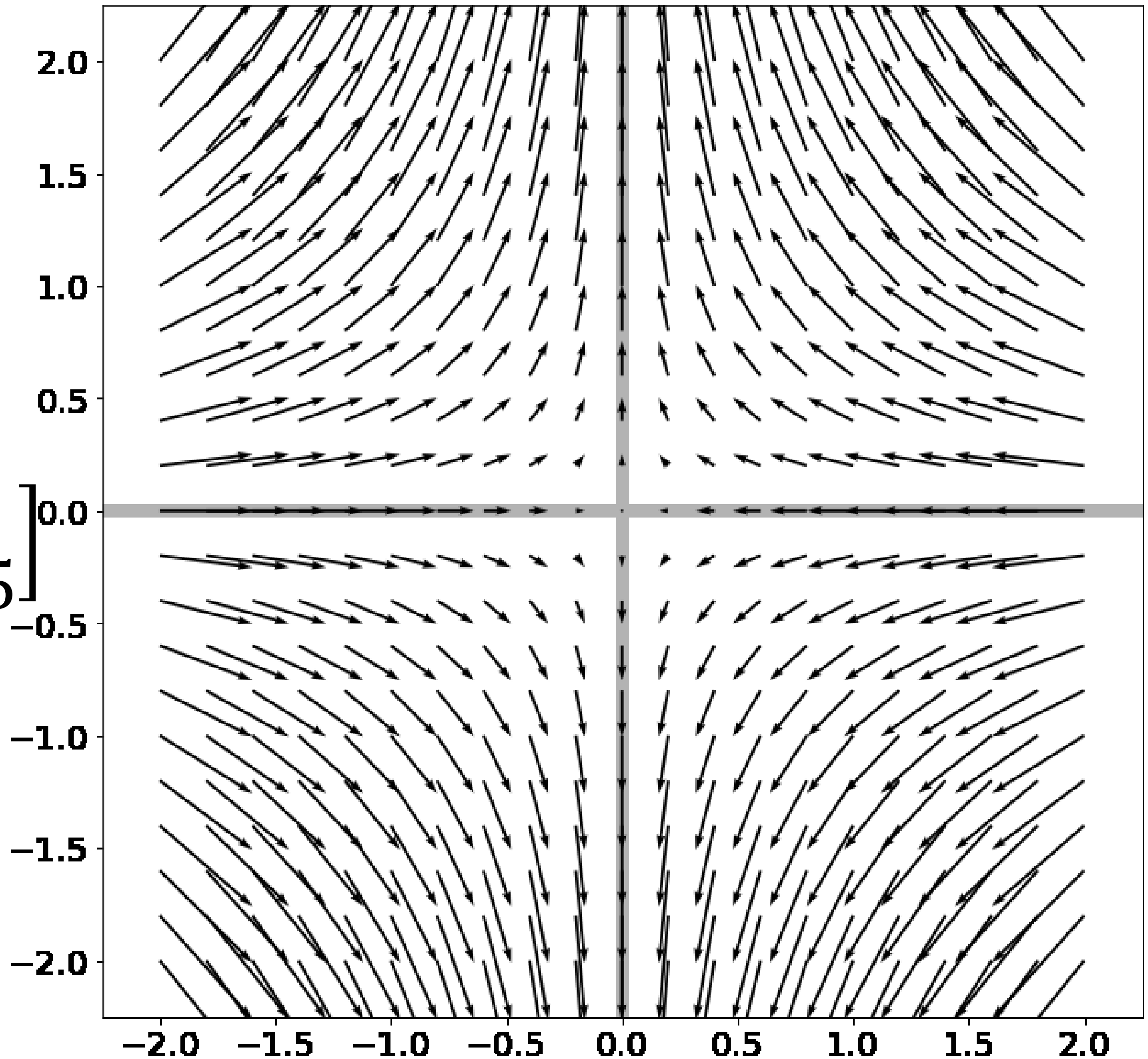
$$A = \begin{bmatrix} 1.1 & 0 \\ 0 & 1.1 \end{bmatrix}$$



Red box – unit square, Blue box – after f(**x**) = **Ax**.

What are the yellow lines and why?

$$\boldsymbol{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 1.25 \end{bmatrix}$$



Now I apply f(**x**) = **Ax** to these points

Pointy-end: **Ax** . Non-Pointy-End: **x**

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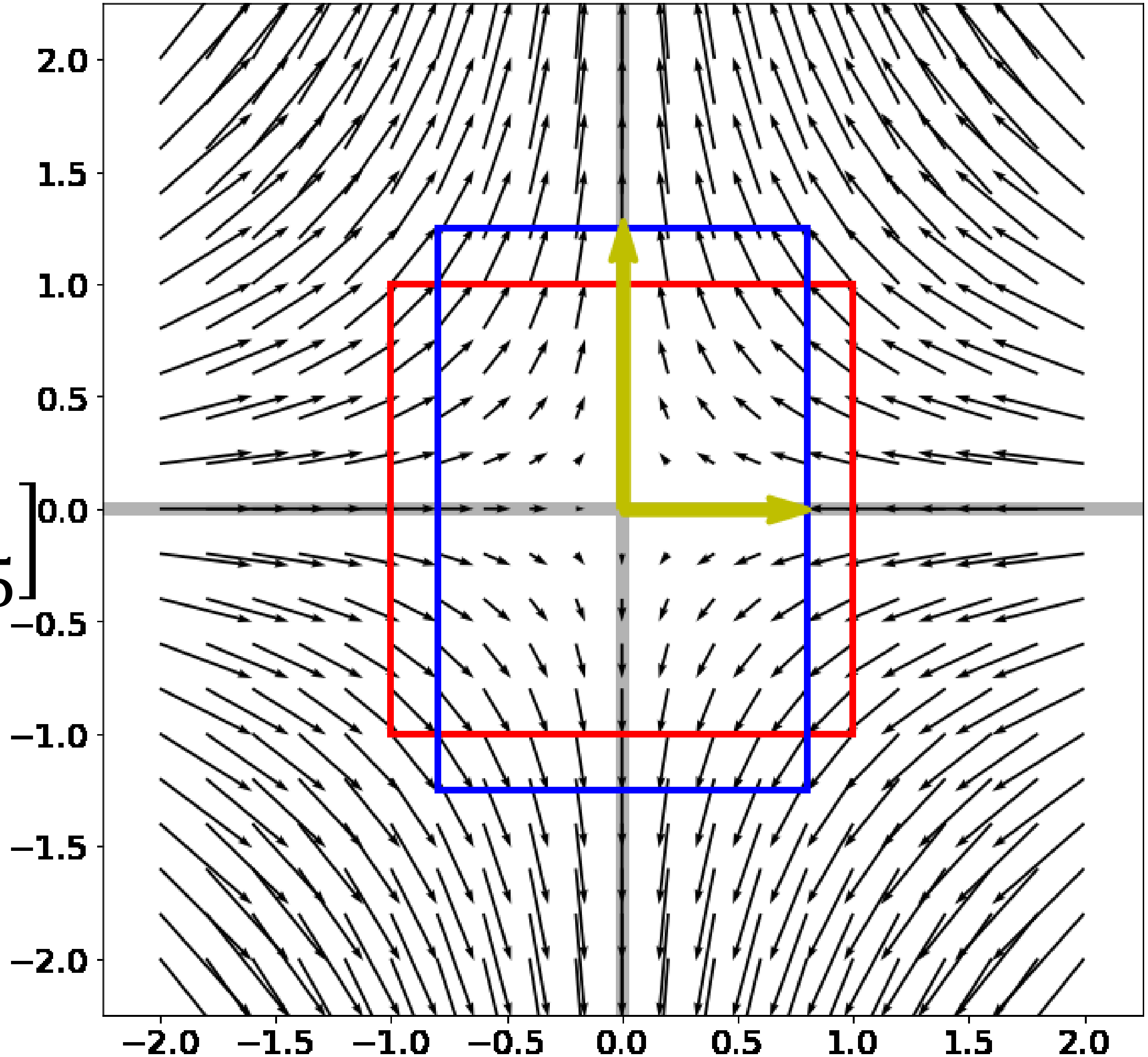
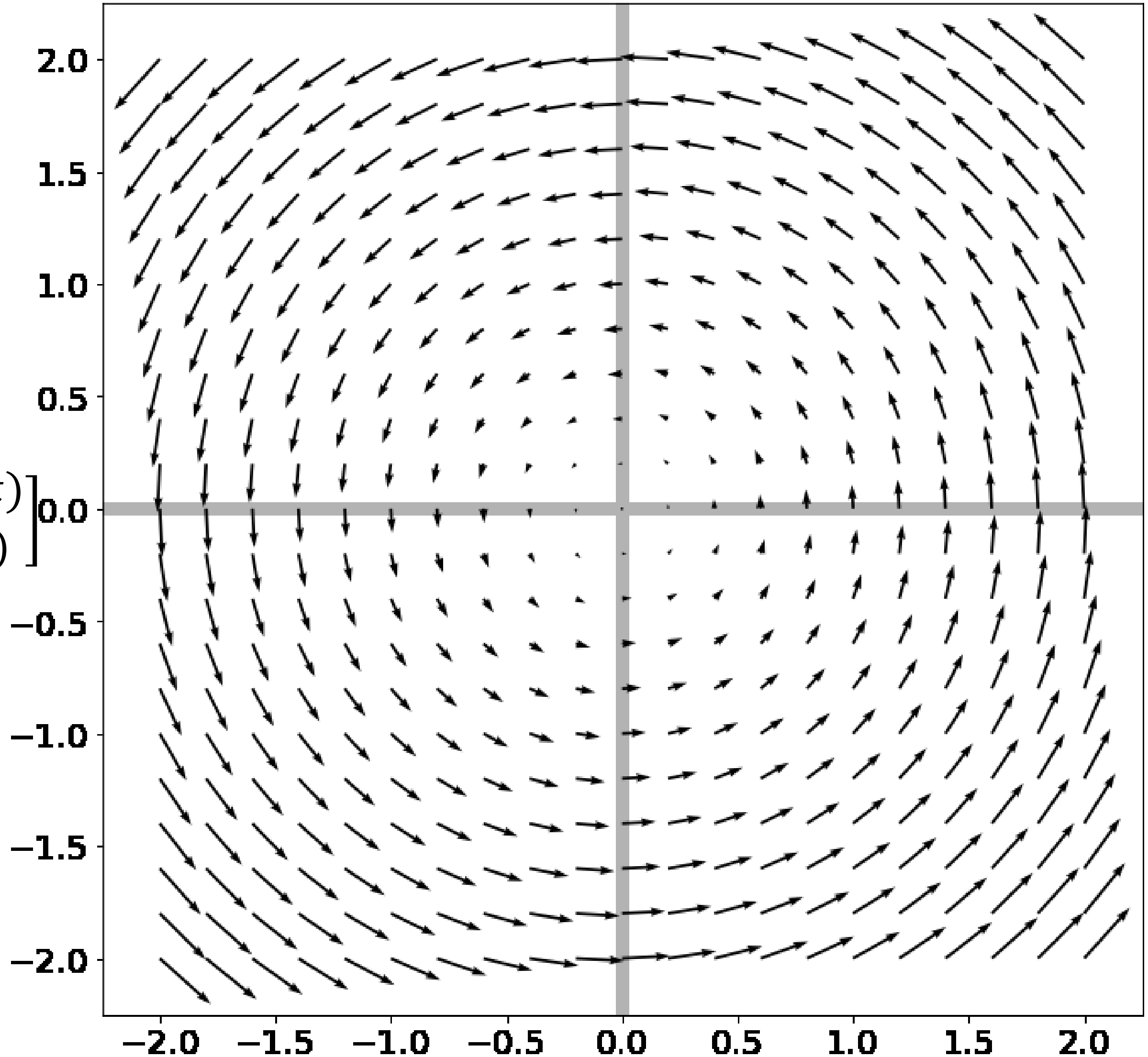


Red box – unit square, Blue box – after f(**x**) = **Ax**.

What are the yellow lines and why?

$$\boldsymbol{A} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$



Red box – unit square, Blue box – after f(**x**) = **Ax**.

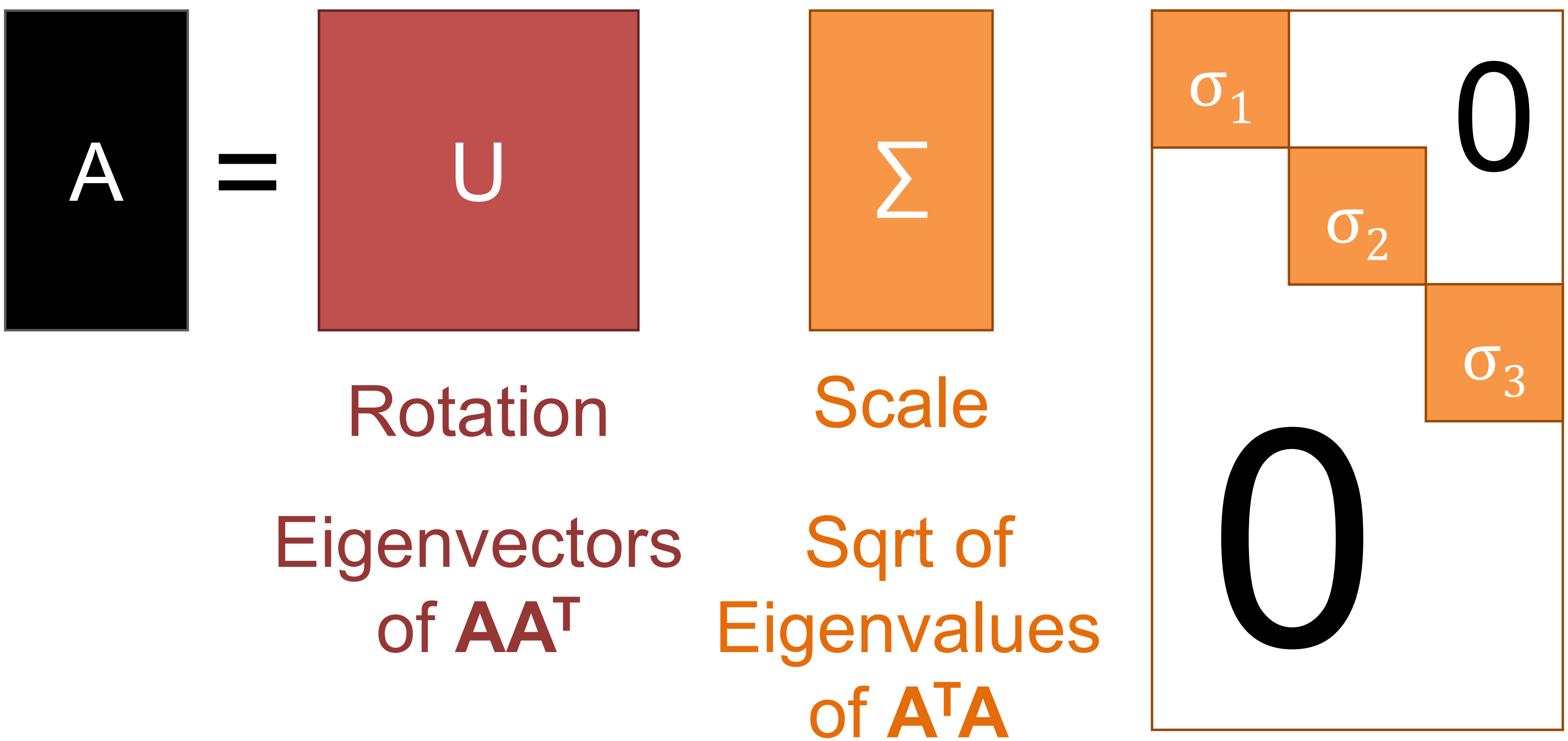
Can we draw any yellow lines?

Eigenvectors of Symmetric Matrices

- Always n mutually orthogonal eigenvectors with n (not necessarily) distinct eigenvalues
- For symmetric $\boldsymbol{A}$, the eigenvector with the largest eigenvalue maximizes $\frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}}$ (smallest/min)
- So for unit vectors (where $\boldsymbol{x}^T\boldsymbol{x} = 1$), that eigenvector maximizes $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$
- A surprisingly large number of optimization problems rely on (max/min)imizing this

The Singular Value Decomposition

Can **always** write a mxn matrix **A** as: $\boldsymbol{A} = \boldsymbol{U \Sigma V^T}$



The Singular Value Decomposition

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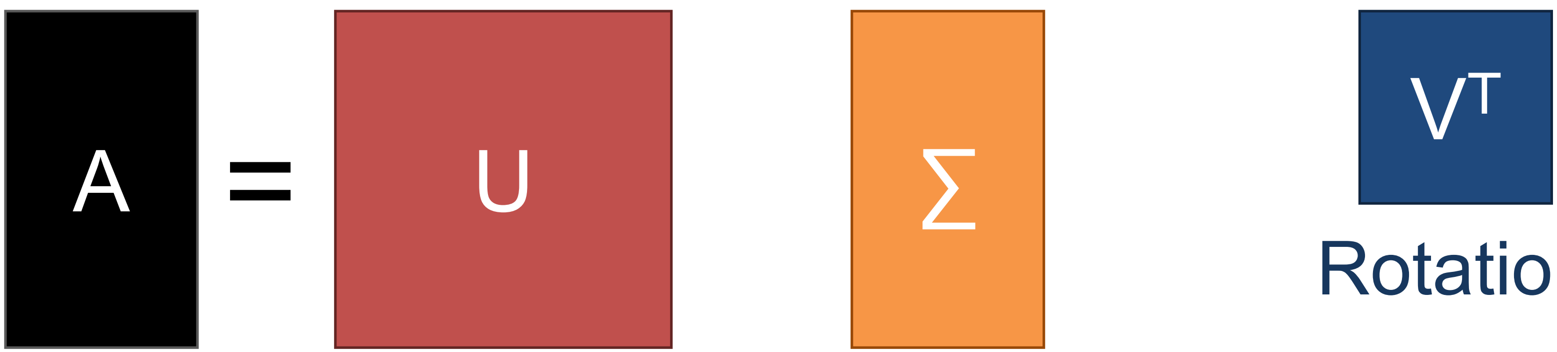


Rotation

Rotation

Scale

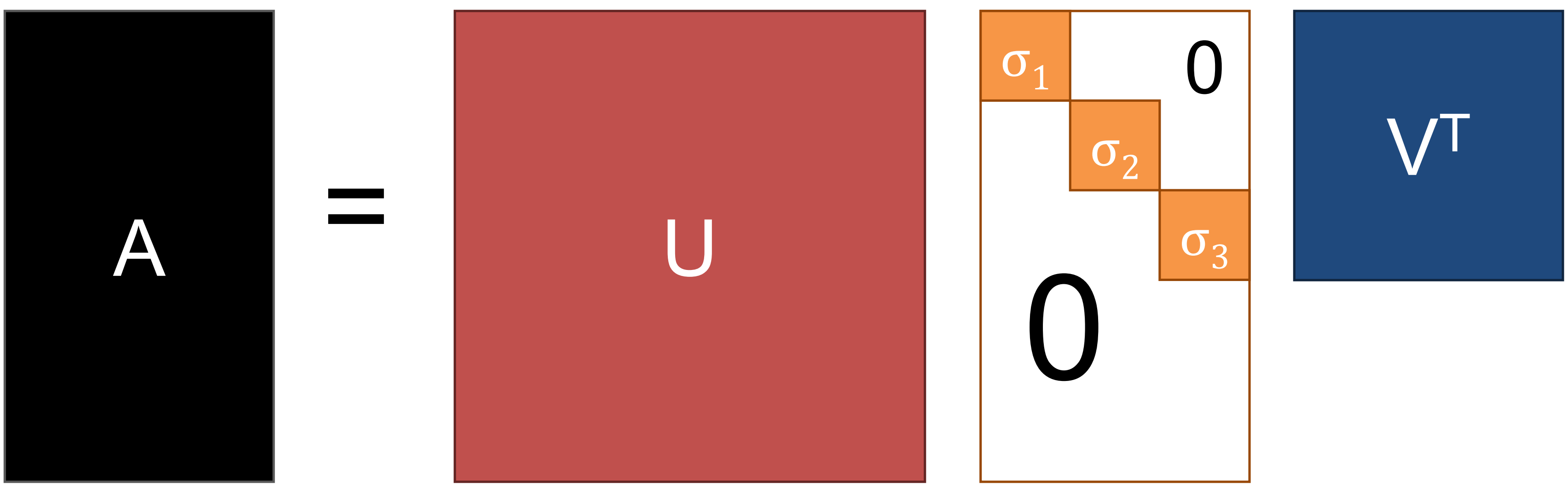
Eigenvectors of $\mathbf{AA^T}$

Sqrt of Eigenvalues of $\mathbf{A^TA}$

Eigenvectors of $\mathbf{A^TA}$

Singular Value Decomposition

- *Every* matrix is a rotation, scaling, and rotation
- Number of non-zero singular values = rank / number of linearly independent vectors
- "Closest" matrix to **A** with a lower rank



Singular Value Decomposition

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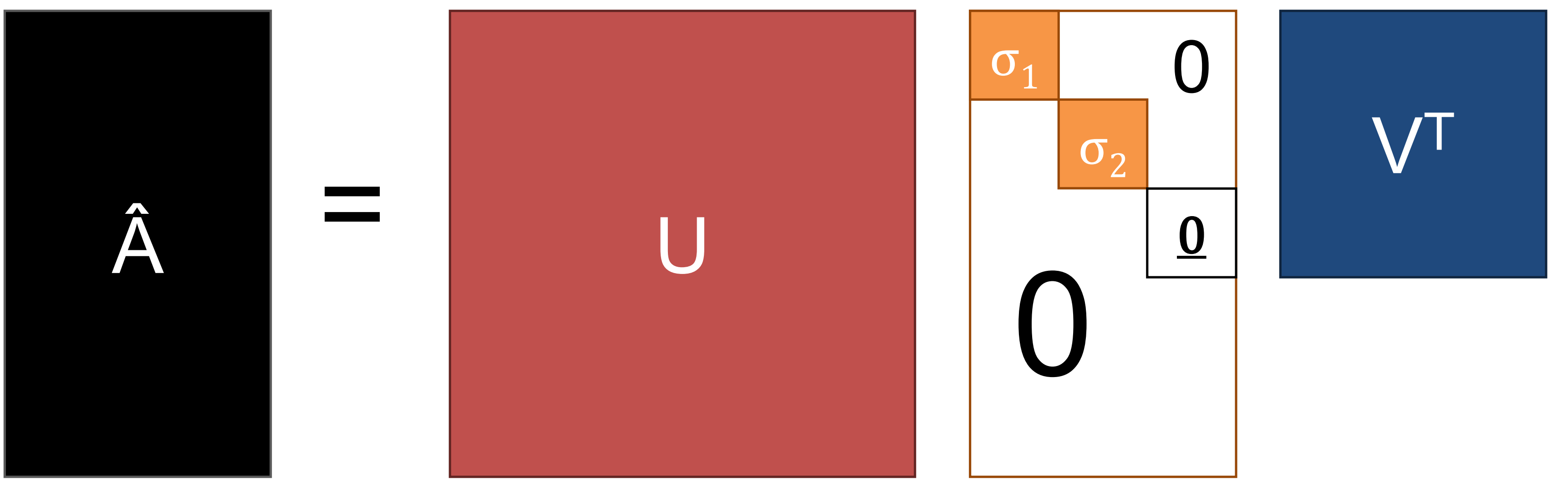
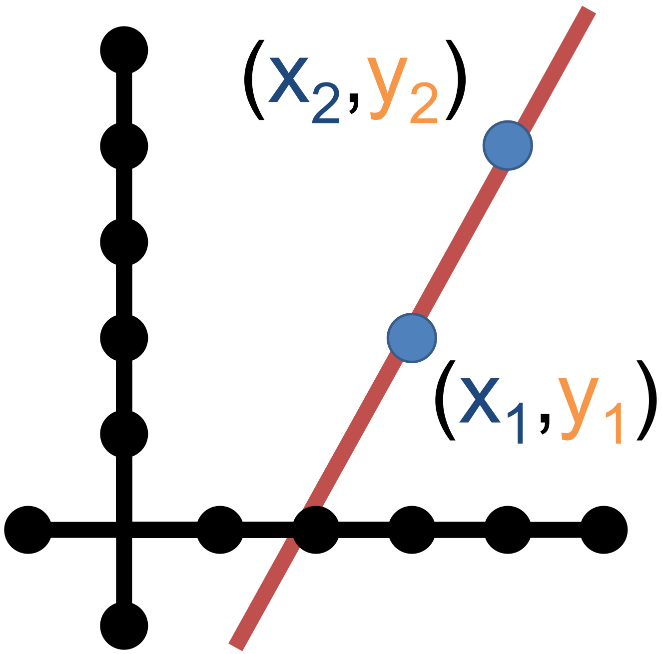


Singular Value Decomposition

- *Every* matrix is a rotation, scaling, and rotation
- Number of non-zero singular values = rank / number of linearly independent vectors
- “Closest” matrix to **A** with a lower rank
- Secretly behind basically many things you do with matrices

Solving Least-Squares



Start with two points (x_i,y_i)

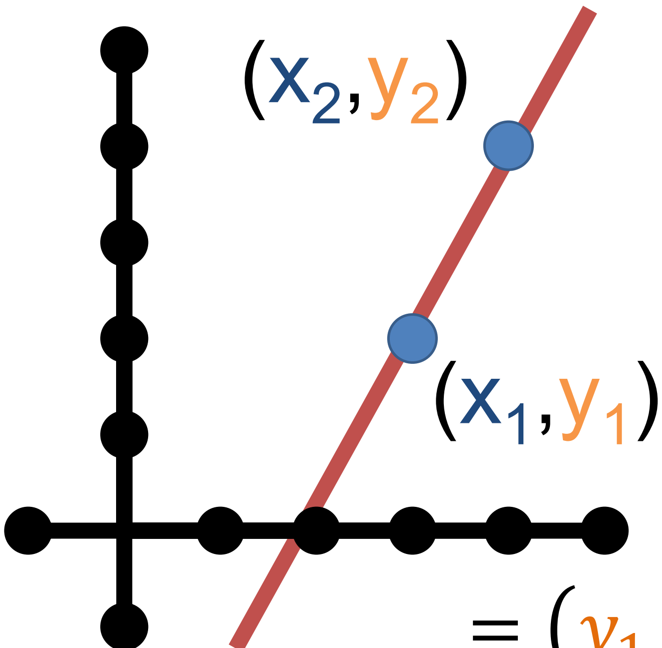
$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{v}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} mx_1 + b \\ mx_2 + b \end{bmatrix}$$

We know how to solve this – invert **A** and find **v** (i.e., (m,b) that fits points)

Solving Least-Squares



Start with two points (x_i, y_i)

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{v}$$

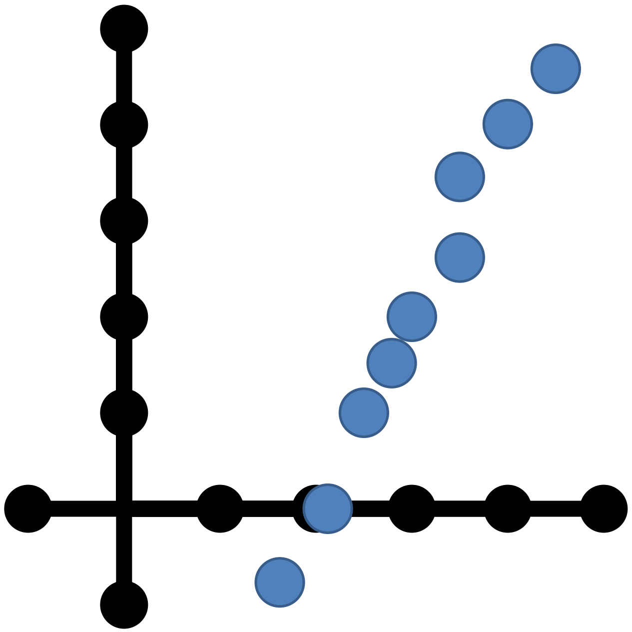
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

$$\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{v}\|^2 = \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} mx_1 + b \\ mx_2 + b \end{bmatrix} \right\|^2$$

$$= \left(y_1 - (mx_1 + b)\right)^2 + \left(y_2 - (mx_2 + b)\right)^2$$

The sum of squared differences between **the actual value of y** and **what the model says y should be.**

Solving Least-Squares



Suppose there are n > 2 points

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{v}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

Compute $||y - Ax||^2$ again

$$||\boldsymbol{y} - \boldsymbol{A}\boldsymbol{v}||^2 = \sum_{i=1}^{n} (y_i - (mx_i + b))^2$$

Solving Least-Squares

Given **y**, **A**, and **v** with **y** = **Av** overdetermined

(**A** tall / more equations than unknowns)

We want to minimize $||\boldsymbol{y} - \boldsymbol{Av}||^2$, or find:

$$\boxed{\arg\min_{\boldsymbol{v}}} ||\boldsymbol{y} - \boldsymbol{Av}||^2$$

(The value of x that makes the expression smallest)

Solution satisfies $\left(\boldsymbol{A}^T\boldsymbol{A}\right)\boldsymbol{v}^* = \boldsymbol{A}^T\boldsymbol{y}$

<u>or</u>

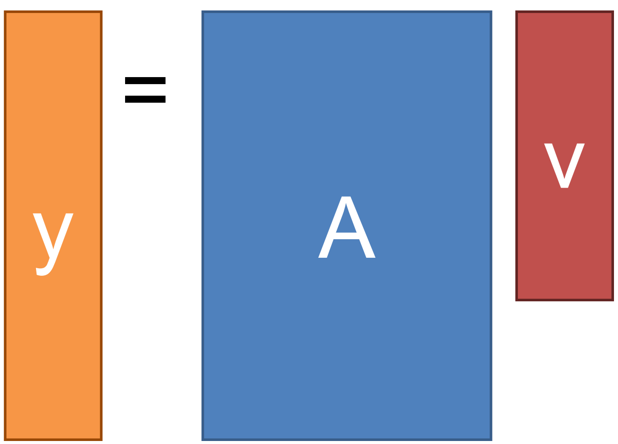
$$\boldsymbol{v}^* = \left(\boldsymbol{A}^T\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T\boldsymbol{y}$$

(Don't actually compute the inverse!)

When is Least-Squares Possible?

Given **y**, **A**, and **v**. Want **y** = **Av**

Want n outputs, have n knobs to fiddle with, every knob is useful if A is full rank.

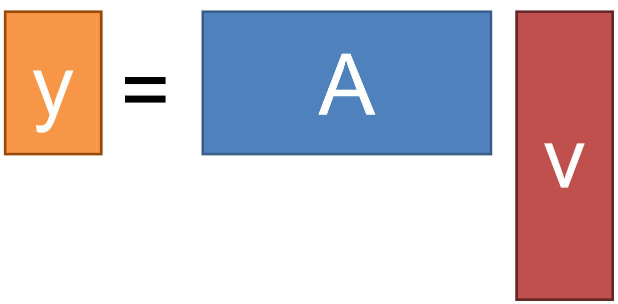


A: rows (outputs) > columns (knobs). Thus can't get precise output you want (not enough knobs). So settle for "closest" knob setting.

When is Least-Squares Possible?

Given **y**, **A**, and **v**. Want **y** = **Av**

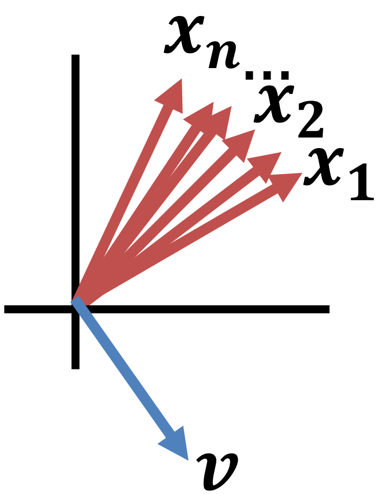
Want n outputs, have n knobs to fiddle with, every knob is useful if A is full rank.



A: columns (knobs) > rows (outputs). Thus, any output can be expressed in infinite ways.

Homogeneous Least-Squares

Given a set of unit vectors (aka directions) $x_1, \ldots, x_n$ and I want vector v that is as orthogonal to all the x_i as possible (for some definition of orthogonal)



Stack x_i into **A**, compute **Av**

$$Av = \begin{bmatrix} - & x_1^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix} v = \begin{bmatrix} x_1^T v \\ \vdots \\ x_n^T v \end{bmatrix}$$

0 if orthog

Compute $\|Av\|^2 = \sum_i^n \left(x_i^T v\right)^2$

Sum of how orthog. v is to each x

Homogeneous Least-Squares

- A lot of times, given a matrix **A** we want to find the **v** that minimizes $\|\boldsymbol{Av}\|^2$.
- I.e., want $\mathbf{v}^* = \arg\min_{\boldsymbol{v}} \|\boldsymbol{Av}\|_2^2$
- What's a trivial solution?
- Set **v = 0** → **Av = 0**
- Exclude this by forcing v to have unit norm

Homogeneous Least-Squares

Let's look at $||\boldsymbol{Av}||_2^2$

$||\boldsymbol{Av}||_2^2 =$ Rewrite as dot product

$||\boldsymbol{Av}||_2^2 = (\mathbf{Av})^{\mathrm{T}}(\mathbf{Av})$ Distribute transpose

$||\boldsymbol{Av}||_2^2 = \boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}}\mathbf{Av} = \mathbf{v}^{\mathbf{T}}\left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)\mathbf{v}$

We want the vector minimizing this quadratic form
Where have we seen this?

Homogeneous Least-Squares

Ubiquitious tool in vision:

$$\arg \min_{\|\boldsymbol{v}\|^2=1} \|\boldsymbol{A}\boldsymbol{v}\|^2$$

→ **(1)** "Smallest"* eigenvector of $\boldsymbol{A}^T\boldsymbol{A}$

(2) **"**Smallest" right singular vector of $\boldsymbol{A}$

For min → max, switch smallest → largest

*Note: $\boldsymbol{A}^T\boldsymbol{A}$ is positive semi-definite so it has all non-negative eigenvalues

Derivatives

Remember derivatives?

Derivative: rate at which a function f(x) changes at a point as well as the direction that increases the function

Given quadratic function f(x)

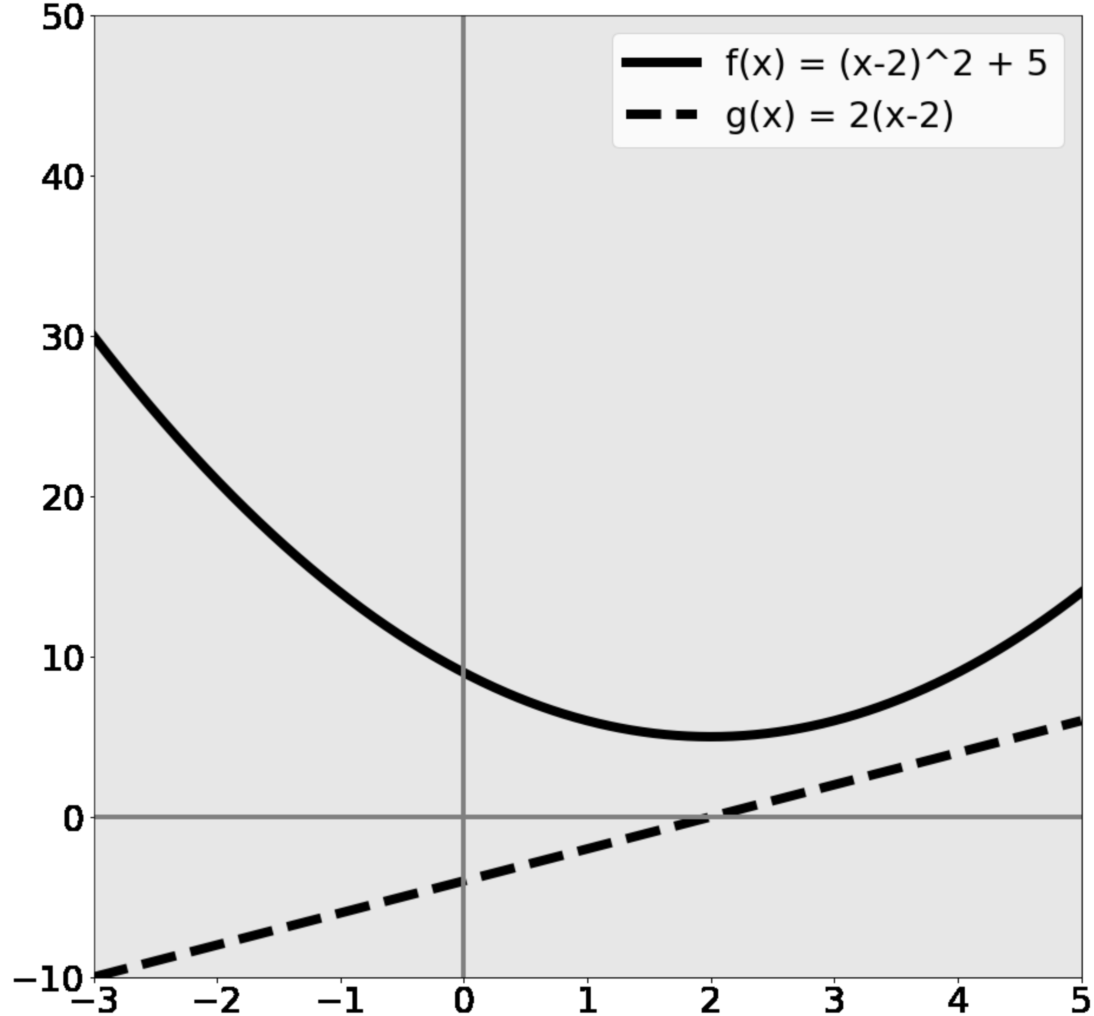
$$f(x, y) = (x - 2)^2 + 5$$

$f(x)$ is function

$$g(x) = f'(x)$$

aka

$$g(x) = \frac{d}{dx} f(x)$$



Given quadratic function f(x)

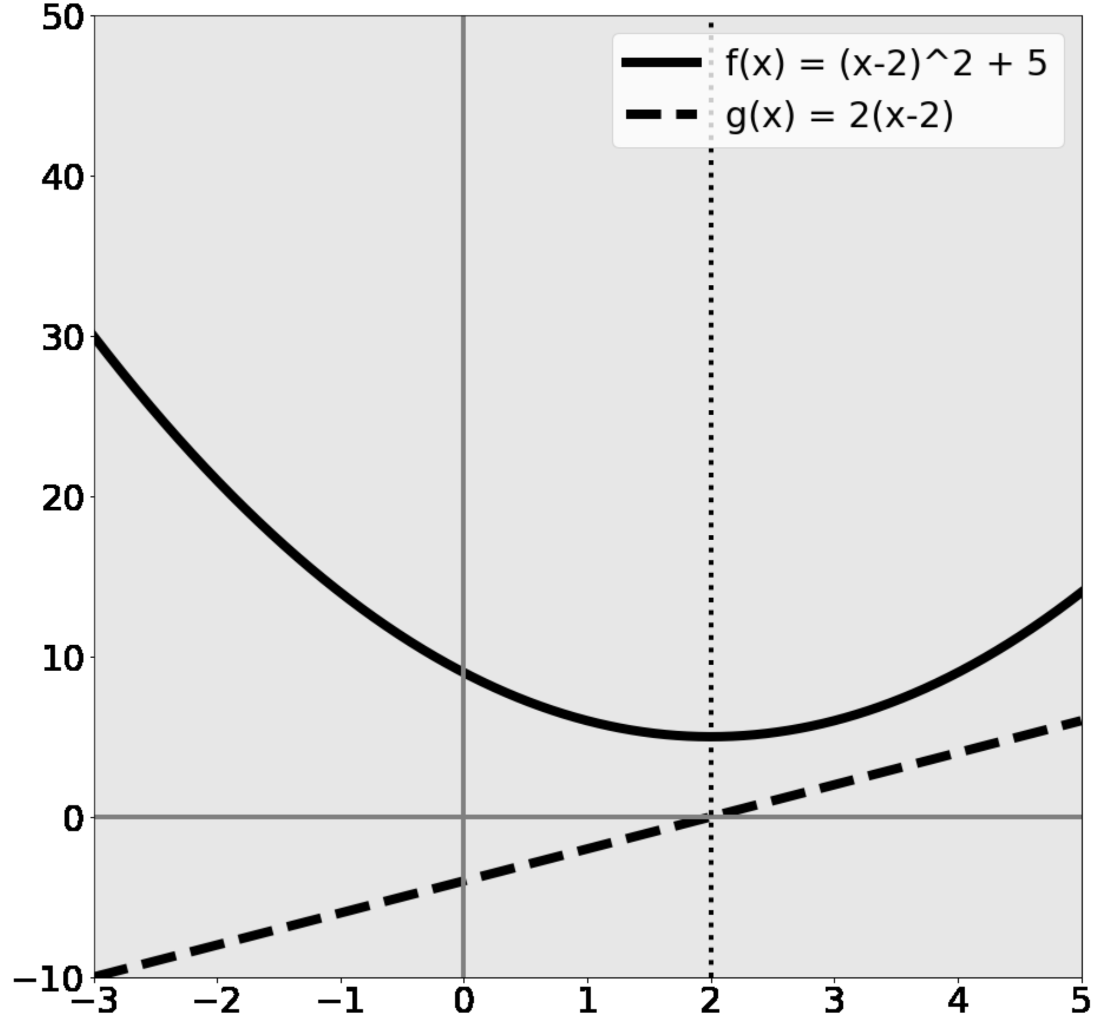
$$f(x, y) = (x-2)^2 + 5$$

What's special about x=2?

$f(x)$ minim. at 2
$g(x) = 0$ at 2

a = minimum of f → $g(a) = 0$

Reverse is ***<u>not true</u>***



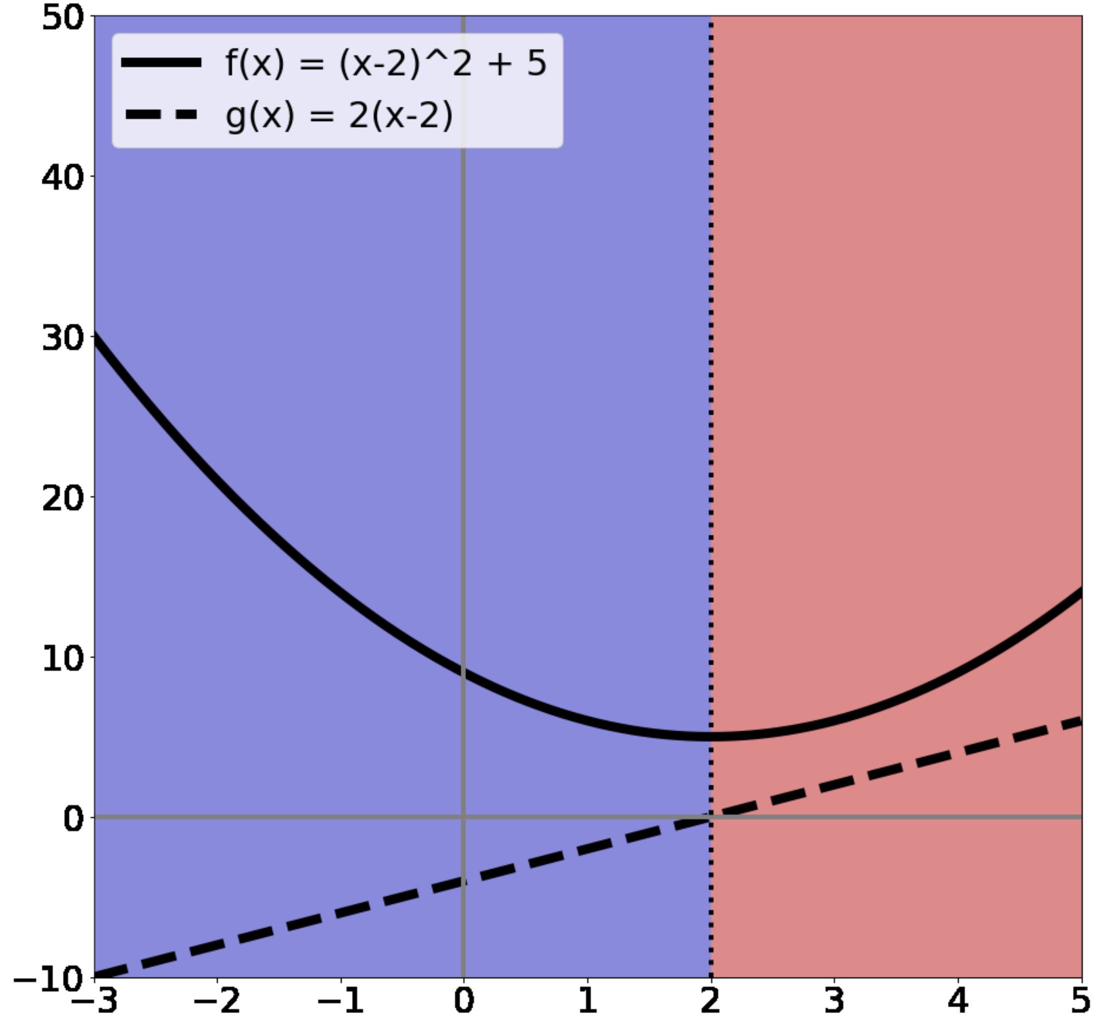
Rates of change

$$f(x,y) = (x-2)^2 + 5$$

Suppose I want to increase f(x) by changing x:

Blue area: move left
Red area: move right

Derivative tells you direction of ascent and rate



What Calculus Should I Know

- Really need intuition
- Need chain rule
- Rest you should look up / use a computer algebra system / use a cookbook
- Partial derivatives (and that's it from multivariable calculus)

Partial Derivatives

- Pretend other variables are constant, take a derivative. That's it.
- Make our function a function of two variables

$$f(x) = (x-2)^2 + 5$$

$$\frac{\partial}{\partial x} f(x) = 2(x-2) * 1 = 2(x-2)$$

$$f_2(x,y) = (x-2)^2 + 5 + \boxed{(y+1)^2}$$

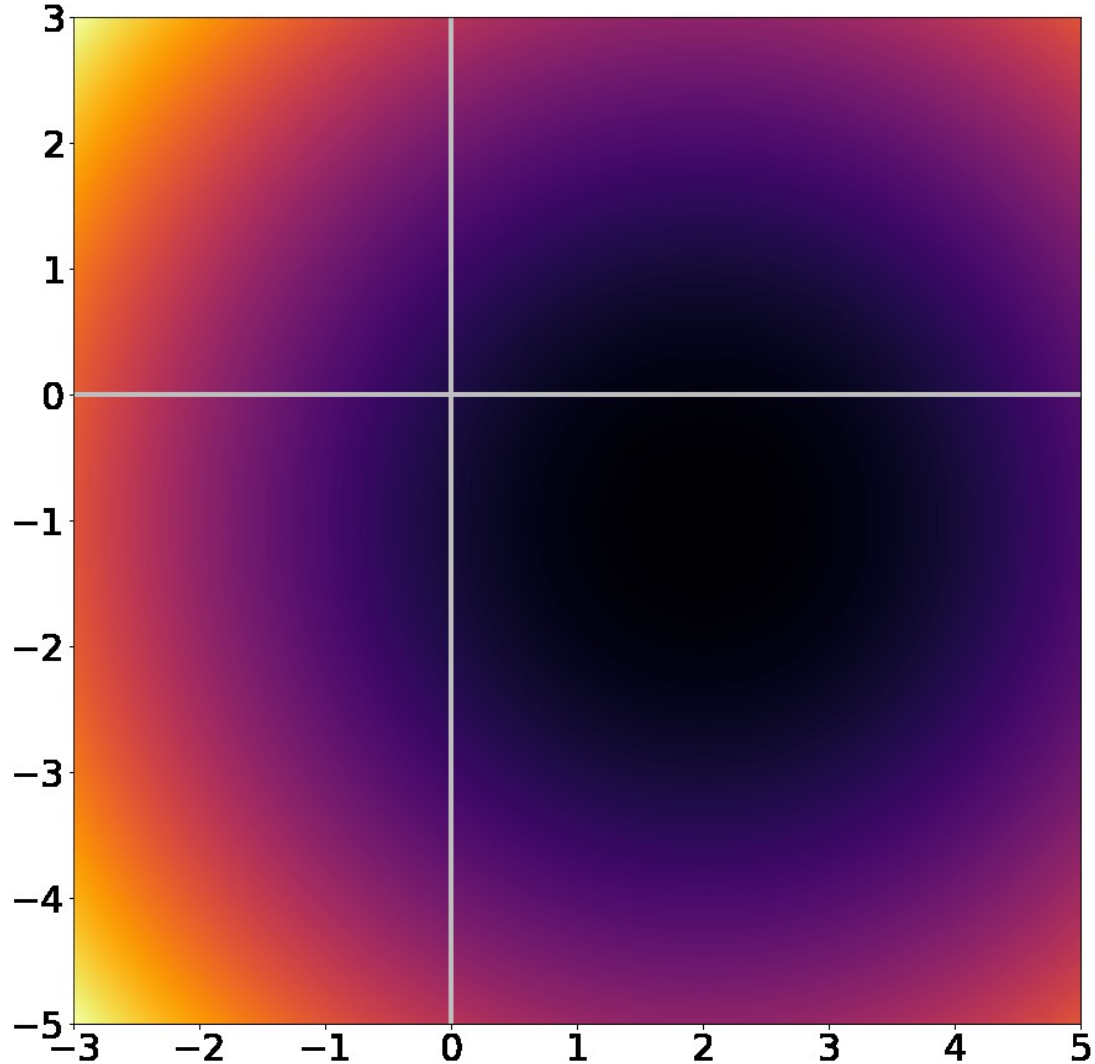
Pretend it's constant → derivative = 0

$$\frac{\partial}{\partial x} f_2(x) = 2(x-2)$$

Zooming Out

$$f_2(x, y) = (x - 2)^2 + 5 + (y + 1)^2$$

Dark = f(x,y) low
Bright = f(x,y) high



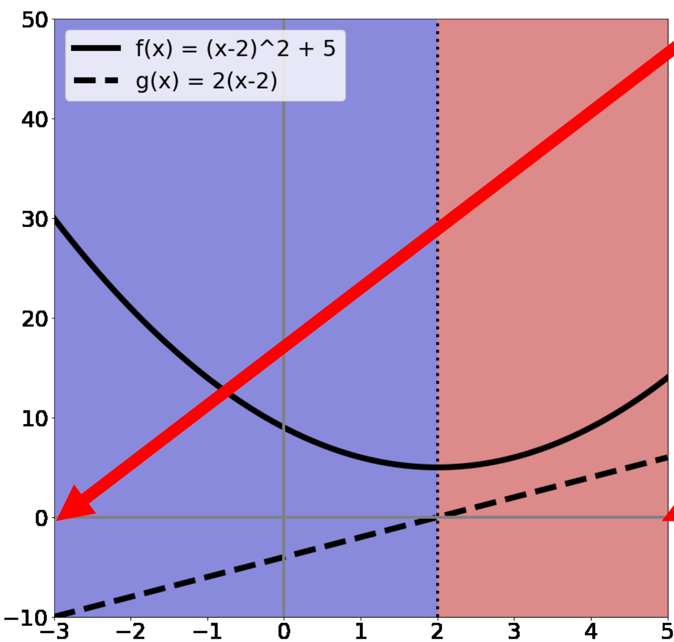
Taking a slice of

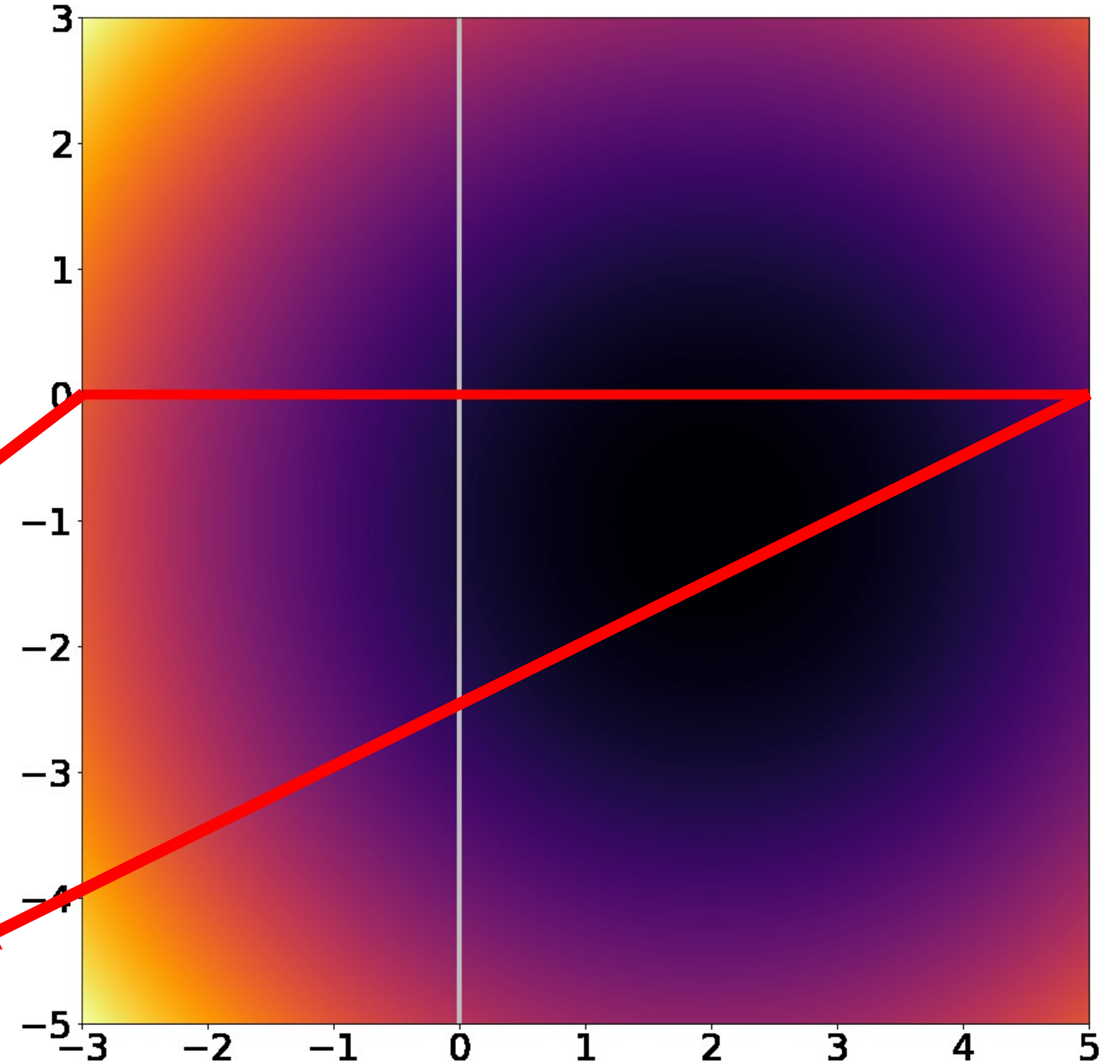
$$f_2(x, y) = (x-2)^2 + 5 + (y+1)^2$$

Slice of y=0 is the function from before:

$$f(x) = (x-2)^2 + 5$$

$$f'(x) = 2(x-2)$$

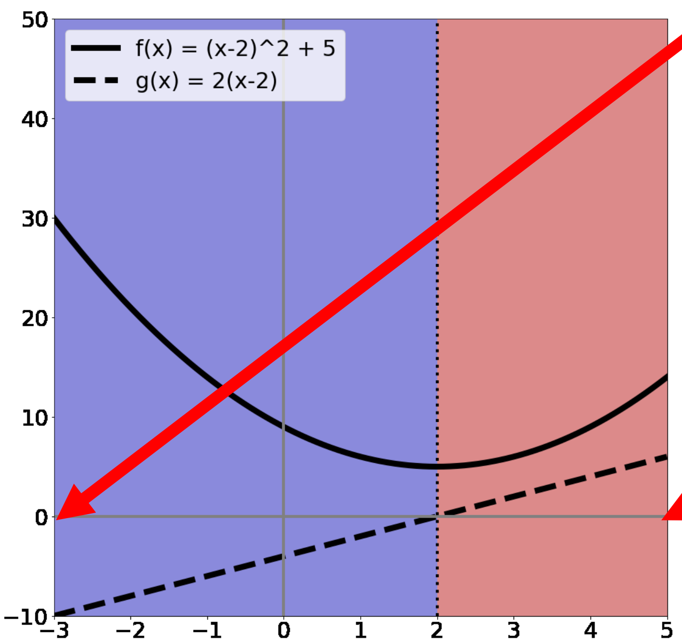


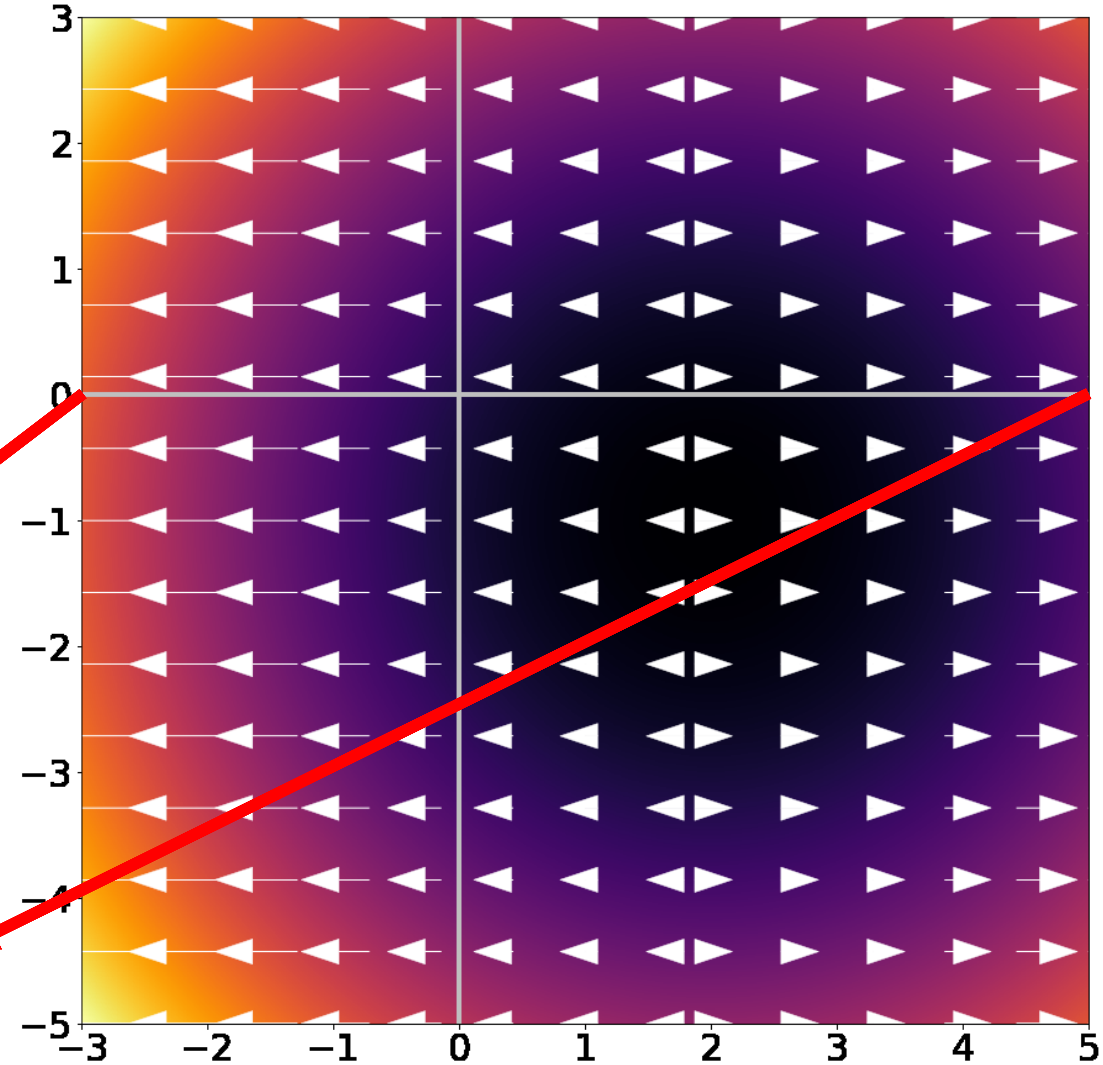


Taking a slice of

$$f_2(x,y) = (x-2)^2 + 5 + (y+1)^2$$

$\frac{\partial}{\partial x} f_2(x,y)$ is rate of change & direction in x dimension





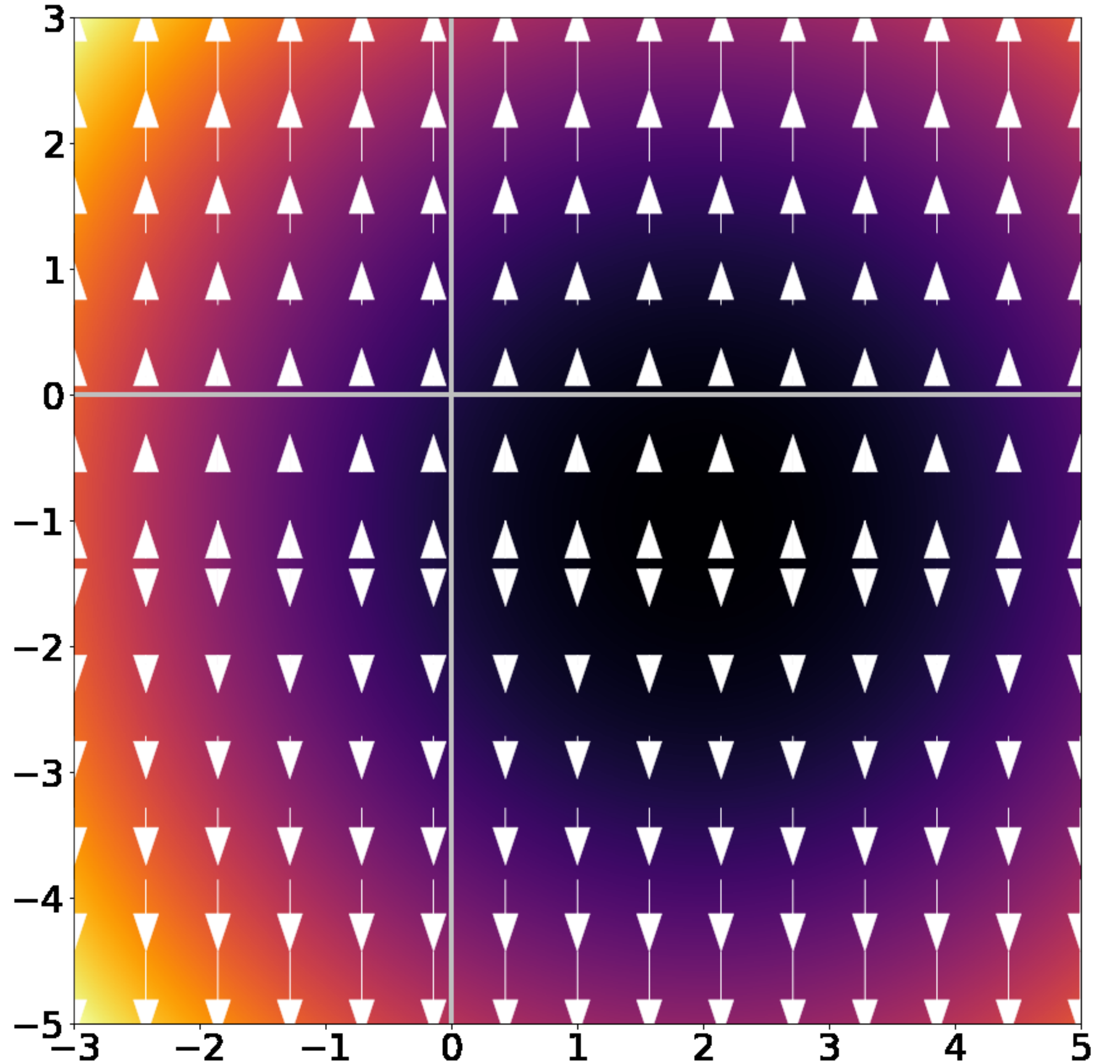
Zooming Out

$$f_2(x,y) = (x-2)^2 + 5 + (y+1)^2$$

$\frac{\partial}{\partial y} f_2(x,y)$ is

$2(y+1)$

and is the rate of change & direction in y dimension



Zooming Out

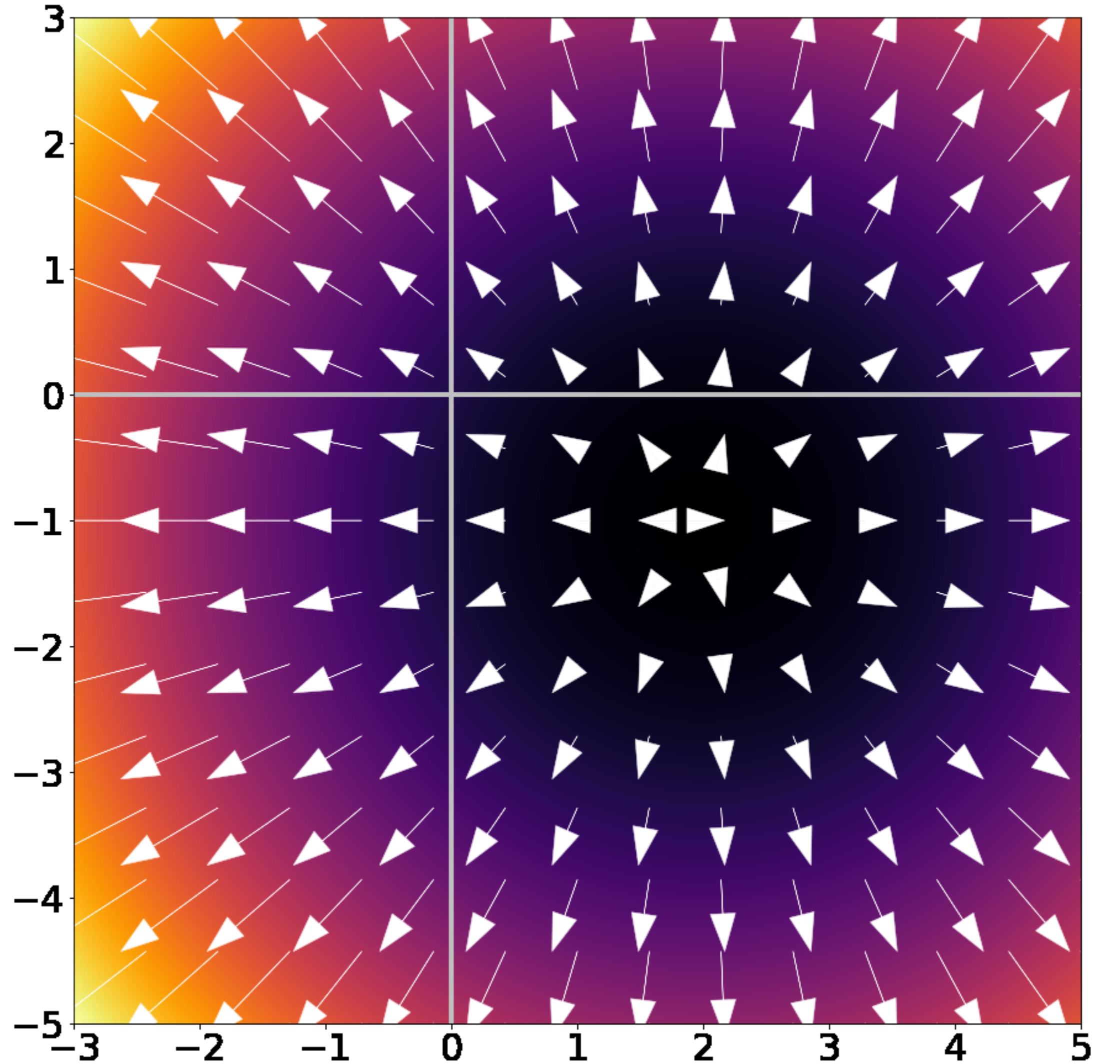
$$f_2(x, y) = (x - 2)^2 + 5 + (y + 1)^2$$

Gradient/Jacobian:
Making a vector of

$$\nabla_f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

gives rate and direction of change.

Arrows point OUT of minimum / basin.



What Should I Know?

- Gradients are simply partial derivatives per-dimension: if $\boldsymbol{x}$ in $f(\boldsymbol{x})$ has n dimensions, $\nabla_f(x)$ has n dimensions
- Gradients point in direction of ascent and tell the rate of ascent
- If a is minimum of $f(\boldsymbol{x}) \rightarrow \nabla_{\mathrm{f}}(\mathrm{a}) = \mathbf{0}$
- Reverse is not true, especially in high-dimensional spaces