# Classic Logic Questions and Answers

CS 188 Section Handout

October 13, 2005

*Note: These answers are not guaranteed to be correct, nor are they the only way to answer these questions. They may not even be totally thorough. They should, however, give you some intuition about how to answer logic questions.*

# 1 Propositional Logic

## 1.1 The semantics of propositional logic

1. Prove that if $(\neg\alpha \vee \beta)$ is valid, then $\alpha \models \beta$.

   **Solution:** $(\neg\alpha \vee \beta)$ means that in all models (truth assignments to the symbols in $\alpha$ and $\beta$, which are sentences like $(A \wedge B)$ or whatever), $(\neg\alpha \vee \beta)$ is true. That is, no matter what I plug in (true or false) for the symbols in this sentence, I'll get true for the whole thing. Now, we decompose...

   By the defined semantics of implication, we know that in all model, either $\alpha$ is false or $\beta$ is true (1). Also, sentences are either true or false given a model. So, for any model $M$ in that set of all models, either $\alpha$ is false or true. If it's true, then we know that $\beta$ is true by (1) above. We have shown that for all models where $\alpha$ is true, $\beta$ is true. This is the definition of $\alpha \models \beta$, so we're done.

2. How many models are there for $Smoke \wedge \neg Fire$?

   **Solution:** A trick question! We don't know how many symbols there are in each model, so we don't know how many possible models there are. But, given that there are $n$ models (e.g. $n = 3$, symbols are $Smoke, Fire, Water$), there are $2^{n-2}$ models for this sentence. One setting for $Smoke$ and $Fire$ together, plus a multiplied 2 for every free symbol.

   If the question is just this, the answer is $\infty$.

3. Prove that $A \Rightarrow (B \wedge A)$ is not valid.

   **Solution:** Any counterexample will do. Set $A$ to true and $B$ to false.

### 1.2 The syntax of propositional logic

1. Show that the distributive rules of $\wedge$ and $\vee$ are in fact true.

   **Solution:** Use a truth table. Write out one of the laws like $(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$ and try all possible (8) combinations of truth values to show that it is always true.

2. Show that any sentence can be converted into a conjunction of clauses of literals.

   **Solution:** We can remove all implications by converting them to disjunctions using the rule $\alpha \Rightarrow \beta \equiv \neg\alpha \vee \beta$, then distribute $\vee$ over any literals connected by $\wedge$ to get a clause in conjunctive normal form. (*this is not a complete proof. You should show why this works in all cases.*)

## 2 First-Order Logic

### 2.1 The semantics of first-order logic

1. Is $\neg\exists x \neg P(x) \Rightarrow \forall x P(x)$ valid? If so, prove it appealing only the definition of validity.

   **Solution:** This prove is a bit complex, so I'm going to write it rather formally. This degree of formality is neither necessary nor particularly desired, but it's instructive. The above sentence in FOL is valid. To prove it, we want to show that for any model (a set of objects and an assignment of true or false to all $P(x)$ for any object x), this sentence is true. We'll do so by appealing only to the definitions of the semantics logical connectives and quantifiers.

   - For any model $M$, either $\neg\exists x \neg P(x)$ is true or false. All sentences are either true or false given a model. This is true of any sentence and model in FOL.
   - If $\neg\exists x \neg P(x)$ is true given $M$, then $\exists x \neg P(x)$ is false.
   - If $\exists x \neg P(x)$ is false, then there are no objects in $M$ such that $\neg P(x)$ is true.
   - Thus, $P(x)$ must be true for all objects in $M$.
   - But this is the definition for the semantics of $\forall x P(x)$, so we have proved that for any model in which $\neg\exists x \neg P(x)$ is true, $\forall x P(x)$ is true.
   - Therefore, $\neg\exists x \neg P(x) \Rightarrow \forall x P(x)$ must be true in all models and therefore valid.

2. Write down a logical sentence such that every model of it has exactly three objects.

   **Solution:** We write a sentence that means there are at least three and at most three objects:

$$
\begin{aligned}
&[\exists x, y, z(x \neq y) \wedge (x \neq z) \wedge (y \neq z)] \wedge \\
&[\forall a, b, c, d((a \neq b) \wedge (b \neq c) \wedge (a \neq c)) \Rightarrow ((a = d) \vee (b = d) \vee (c = d))]
\end{aligned}
$$

3. Constuct a model for which the following is false: $[\forall x \exists y Q(x, y)] \Rightarrow [\exists y \forall x Q(x, y)]$

**Solution:** We can introduce any objects we want, and assign true or false to $Q$ in any way we want while constructing our model. One solution is to give two objects and make $Q$ the identity predicate (true for $Q(a, a)$ and false for $Q(a, b)$ with $a \neq b$).

Let objects be *cat* and *dog*.

| Q(cat, cat) | Q(cat, dog) | Q(dog,cat) | Q(dog, dog) |
|---|---|---|---|
| T | F | F | T |

Then, $\forall x \exists y Q(x, y)$ is true because every object is the same as something (itself), but $\exists y \forall x Q(x, y)$ is false because no object is the same as everything.

## 2.2 Translating English into first order logic

1. Every gal in Constantinople lives in Istanbul, not Constantinople.[1]

   **Solution:** Let *Gal*, *Lives* and *In* be predicates and *Istanbul* and *Constantinople* be constants. Then we have:

$$\forall x \, [Gal(x) \wedge In(x, Constantinople)] \Rightarrow [Lives(x, Istanbul) \wedge \neg Lives(x, Constantinople)] \tag{1}$$

2. Every new beginning comes from some other beginning end.[2]

   **Solution:** We need a predicate *Beginning* that tests if something is a beginning, a predicate *ComesFrom* and a function *end* that maps from objects to their ends.

$$\forall x \exists y Beginning(x) \Rightarrow [Beginning(y) \wedge ComesFrom(x, end(y))] \tag{2}$$

3. How wonderful life is while you're in the world.[3]

   **Solution:** This one's less intuitive to translate. It helps to introduce a notion of time and a predicate $<$ that returns true if one time is less than another. So, let's have predicates $Time$, $<$, $Wonderful$, and $In$ and constants $Life$, $World$, and $You$. The peculiar thing is that we'll have all of these predicates take a time along with their expected variables. We're going to create a sentence that more literally means *life is wondeful while you're in the world*.

$$\forall t_1, t_2, t_3 \, [In(World, You, t_2) \wedge In(World, You, t_3) \wedge (t_2 < t_1) \wedge (t_1 < t_3)] \Rightarrow Wonderful(Life, t_1)$$

   Note that this implies that if you're in the world at two times, you are in the world at all times in between. To really complete the question, we should write a second sentence that expresses this fact.

[1] They Might Be Giants, *Istanbul*
[2] Semisonic, *Closing Time*
[3] Elton John, *Your Song*