Math 55 First Midterm 24 Sep 2007

NAME (1 pt): ________________________________

TA (1 pt): ________________________________

Name of Neighbor to your left (1 pt): ________________________________

Name of Neighbor to your right (1 pt): ________________________________

Instructions: This is a closed book, closed calculator, closed computer, closed network, open brain exam. You are allowed one page of notes (double sided) that can be read without a magnifying glass.

You get one point each for filling in the 4 lines at the top of this page. All other questions are worth 20 points.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

1	
2	
3	
Total	

[1]

Question 1) (20 pts) Determine whether or not the following proposition is a tautology. $a \downarrow b$ is the same as $\neg(a \vee b)$.

$$(r \vee (p \downarrow q)) \rightarrow (\neg r \rightarrow \neg q)$$

If it is, prove it using rules for simplifying logical expressions, not using a truth table. If not, give a counterexample. Indicate what you are doing at each step (i.e. you don't need to use the Latin names of inference rules, just convince us that you know what you're doing).

Answer:

$$(r \vee (p \downarrow q)) \rightarrow (\neg r \rightarrow \neg q)$$

$$\Leftrightarrow \textit{(definition of } \downarrow\textit{)}$$

$$(r \vee \neg(p \vee q)) \rightarrow (\neg r \rightarrow \neg q)$$

$$\Leftrightarrow \textit{(definition of } \rightarrow \textit{ and double negative)}$$

$$(r \vee \neg(p \vee q)) \rightarrow (r \vee \neg q)$$

$$\Leftrightarrow \textit{(definition of } \rightarrow\textit{)}$$

$$\neg(r \vee \neg(p \vee q)) \vee (r \vee \neg q)$$

$$\Leftrightarrow \textit{(DeMorgan's Law and double negative)}$$

$$(\neg r \wedge (p \vee q)) \vee (r \vee \neg q)$$

$$\Leftrightarrow \textit{(distributivity)}$$

$$((\neg r \wedge p) \vee (\neg r \wedge q)) \vee (r \vee \neg q)$$

$$\Leftrightarrow \textit{(DeMorgan's Law and double negative)}$$

$$((\neg r \wedge p) \vee (\neg r \wedge q)) \vee \neg(\neg r \wedge q)$$

$$\Leftrightarrow \textit{(associativity and } x \vee \neg x = T\textit{, where } x = \neg r \wedge q\textit{)}$$

$$(\neg r \wedge p) \vee T$$

$$\Leftrightarrow \textit{(anything } \vee T = T\textit{)}$$

$$T$$

Question 1) (20 pts) Determine whether or not the following proposition is a tautology. $a \downarrow b$ is the same as $\neg(a \vee b)$.

$$((s \downarrow t) \vee \neg v) \rightarrow (v \rightarrow \neg t)$$

If it is, prove it using rules for simplifying logical expressions, not using a truth table. If not, give a counterexample. Indicate what you are doing at each step (i.e. you don't need to use the Latin names of inference rules, just convince us that you know what you're doing).

Answer:

$$((s \downarrow t) \vee \neg v) \rightarrow (v \rightarrow \neg t)$$

$$\Leftrightarrow \textit{(definition of } \downarrow\textit{)}$$

$$(\neg(s \vee t) \vee \neg v) \rightarrow (v \rightarrow \neg t)$$

$$\Leftrightarrow \textit{(definition of } \rightarrow\textit{)}$$

$$(\neg(s \vee t) \vee \neg v) \rightarrow (\neg v \vee \neg t)$$

$$\Leftrightarrow \textit{(definition of } \rightarrow\textit{)}$$

$$\neg(\neg(s \vee t) \vee \neg v) \vee (\neg v \vee \neg t)$$

$$\Leftrightarrow \textit{(DeMorgan's Law and double negative)}$$

$$((s \vee t) \wedge v) \vee (\neg v \vee \neg t)$$

$$\Leftrightarrow \textit{(distributivity)}$$

$$((s \wedge v) \vee (t \wedge v)) \vee (\neg v \vee \neg t)$$

$$\Leftrightarrow \textit{(DeMorgan's Law)}$$

$$((s \wedge v) \vee (t \wedge v)) \vee \neg(v \wedge t)$$

$$\Leftrightarrow \textit{(associativity, commutativity, and } x \vee \neg x = T\textit{, where } x = t \wedge v\textit{)}$$

$$(s \wedge v) \vee T$$

$$\Leftrightarrow \textit{(anything } \vee T = T\textit{)}$$

$$T$$

Question 2. (20 points) Classify each function f as one-to-one, onto, both, or neither. Justify your answers.

2.1) (5 points) Let $\mathbb{Q}^+$ denote the set of positive rational numbers and $\mathbb{Z}$ denote the set of integers. A rational number $r \in \mathbb{Q}^+$ can be written in exactly one way as $r = p/q$ where p and q are positive integers and have no common divisor bigger than 1. We define the function $f : \mathbb{Q}^+ \to \mathbb{Z}$ as follows: if $r \in \mathbb{Q}^+$ and $r = p/q$ as above, then $f(p/q) = p - q$. For example $f(7/2) = 5$.

Answer: f is onto: If $z \geq 0$, $f((z+1)/1) = z$, since $z+1$ and 1 are positive with no common factors bigger than 1. If $z < 0$, $f(1/(1-z)) = z$, since 1 and $1-z$ are positive with no common factors bigger than 1. f is not one-to-one, since $f(3/1) = f(5/3) = 2$.

2.2) (5 points) Let $\mathbb{Q}$ denote the set of rational numbers and $\mathbb{R}$ the set of real numbers. Let $A = \{r : \ r \in \mathbb{Q}$ and $0 \leq r \leq 1\}$ and $B = \{r : \ r \in \mathbb{R}$ and $0 \leq r \leq 1\}$. Let $f : A \to B$ be defined by $f(x) = 2^x - 1$.

Answer: f is one-to-one because it is a strictly increasing function of x. f is not onto because A is countable and B is not countable, so there cannot be a bijection between A and B.

2.3) (5 points) Let $h(x) = \lceil \frac{9x}{4} \rceil - \lfloor \frac{5x}{4} \rfloor - 2$, $g(x) = x + 1$, and $f : \{0, 1, 2, 3\} \to \{0, 1, 2, 3\}$ be defined by $f = h \circ g$.

Answer: f maps 0,1,2,3 to 0,1,2,2, so it is neither one-to-one nor onto.

2.4) (5 points) Let B be the set of all bit strings of length 8, and $B \times B$ be the Cartesian product, i.e. the set of all pairs of bit strings of length 8. If $b \in B$, let b_i denote the i-th bit of b for $i = 1, 2, ..., 8$. We interpret 0 as True and 1 as False, so we can write logical expressions like $b_i \oplus b_{i+1}$, where $\oplus$ is exclusive-or.

We define the function $f : B \times B \to B \times B$ as follows: $f(w, x) = (y, z)$ where $y_i = w_i$ and $z_i = w_i \ \oplus x_i$.

Answer: f is one-to-one: Suppose $f(w, x) = f(\hat{w}, \hat{x}) = (y, z)$. Then $w = y = \hat{w}$ and

$$x = x \oplus F = x \oplus (w \oplus w) = (x \oplus w) \oplus w = z \oplus y = (\hat{x} \oplus \ \hat{w}) \oplus \hat{w} = \hat{x} \oplus (\hat{w} \oplus \hat{w}) = \hat{x} \oplus F = \hat{x}$$

f is onto: given (y, z), choose $w = y$, and $x = y \oplus z$, so $w \oplus x = y \oplus (y \oplus z) = (y \oplus y) \oplus z = F \oplus z = z$.

Question 2. (20 points) Classify each function g as one-to-one, onto, both, or neither. Justify your answers.

2.1) (5 points) Let $\mathbb{Q}^+$ denote the set of positive rational numbers and $\mathbb{Z}$ denote the set of integers. A rational number $x \in \mathbb{Q}^+$ can be written in exactly one way as $x = r/s$ where r and s are positive integers and have no common divisor bigger than 1. We define the function $g : \mathbb{Q}^+ \to \mathbb{Z}$ as follows: if $x \in \mathbb{Q}^+$ and $x = r/s$ as above, then $g(r/s) = s - r$. For example $g(5/3) = -2$.

Answer: g is onto: If $z \geq 0$, $g(1/(z+1)) = z$, since $z+1$ and 1 are positive with no common factors bigger than 1. If $z < 0$, $g((1-z)/1) = z$, since 1 and $1-z$ are positive with no common factors bigger than 1. g is not one-to-one, since $g(3/1) = g(5/3) = -2$.

2.2) (5 points) Let $\mathbb{Q}$ denote the set of rational numbers and $\mathbb{R}$ the set of real numbers. Let $A = \{r : r \in \mathbb{Q} \text{ and } 1 \leq r \leq 4\}$ and $B = \{r : r \in \mathbb{R} \text{ and } 1 \leq r \leq 2\}$. Let $g : A \to B$ be defined by $g(x) = \sqrt{x}$.

Answer: g is one-to-one because it is a strictly increasing function of x. g is not onto because A is countable and B is not countable, so there cannot be a bijection between A and B.

2.3) (5 points) Let $q(y) = \lceil \frac{9}{4}y \rceil - \lfloor \frac{5}{4}y \rfloor - 2$, $r(y) = y + 1$, and $g : \{-1, 0, 1, 2\} \to \{-1, 0, 1, 2\}$ be defined by $g = r \circ q$.

Answer: g maps -1,0,1,2 to -1,-1,1,2 so it is neither one-to-one nor onto.

2.4) (5 points) Let S be the set of all bit strings of length 10, and $S \times S$ be the Cartesian product, i.e. the set of all pairs of bit strings of length 10. If $s \in S$, let s_i denote the i-th bit of s for $i = 1, 2, ..., 10$. We interpret 0 as True and 1 as False, so we can write logical expressions like $s_i \oplus s_{i+1}$, where $\oplus$ is exclusive-or.

We define the function $g : S \times S \to S \times S$ as follows: $g(z, w) = (y, x)$ where $x_i = z_i$ and $y_i = w_i \oplus z_i$.

Answer: g is one-to-one: Suppose $g(z, w) = g(\hat{z}, \hat{w}) = (y, x)$. Then $z = x = \hat{z}$ and

$$w = w \oplus F = w \oplus (z \oplus z) = (w \oplus z) \oplus z = y \oplus x = (\hat{w} \oplus \hat{z}) \oplus \hat{z} = \hat{w} \oplus (\hat{z} \oplus \hat{z}) = \hat{w} \oplus F = \hat{w}$$

g is onto: given (y, x), choose $z = x$, and $w = y \oplus x$, so $w \oplus z = (y \oplus x) \oplus x = y \oplus (x \oplus x) = y \oplus F = y$.

Question 3 (20 points) In the expressions below $\log x$ means $\log_2 x$. Show your work to get full credit.

3.1) (6 points) Find the smallest integer n such that

$$f(x) = \sqrt{487x^7(\log x)^3 + 151x^4\sqrt[3]{x^{10}}(\log\log x)^7 - 3x\sqrt{x^{11}} - 4!}$$

is $O(x^{n/6})$. Note that the exponent is $n/6$.

Answer: Of the terms being added under the square root sign, $O(x^7(\log x)^3)$, $O(x^{\frac{22}{3}}(\log\log x)^7)$, $O(x^{\frac{13}{2}})$, *and* $O(1)$, *the second has the largest power of* x. *Since* $\log x = O(x^a)$ *for any* $a > 0$, *the second term is largest, and the sum is* $O(x^{\frac{22}{3}}(\log\log x)^7)$. *Taking the final square root yields* $O(x^{\frac{11}{3}}(\log\log x)^{\frac{7}{2}})$. *This is not* $O(x^{\frac{22}{6}})$ *but is* $O(x^{\frac{23}{6}})$ *for similar reasons, so* $n = 23$.

3.2) (7 points) The notation $\prod_{i=1}^{k} p_i$ means the product $p_1 \cdot p_2 \cdots p_k$. Find a simple accurate function $h(n)$ such that $\prod_{i=1}^{n^2} 5^{3i+7\log i}$ is $O(h(n))$.

Answer: $\prod_{i=1}^{n^2} 5^{3i+7\log i} \leq \prod_{i=1}^{n^2} 5^{3i+7\log n^2} = \prod_{i=1}^{n^2} 5^{3i}5^{7\log n^2} = 5^{7n^2\log n^2}\prod_{i=1}^{n^2} 5^{3i} = n^{14n^2\log 5}\prod_{i=1}^{n^2} 5^{3i} = n^{14n^2\log 5}5^{\sum_{i=1}^{n^2} 3i} = n^{14n^2\log 5}5^{3n^2(n^2+1)/2}$.

3.3) (7 points) Find a simple accurate function $h(x)$ such that
$g(x) = \log(\log((x!)^{(x!)})) + x(\log x)^{1/2}$ is $O(h(x))$.

Answer: We simplify the first term to get $\log\log(x!)^{(x!)} = \log((x!)\log(x!)) = \log(x!) + \log\log(x!)$. *The first term is arbitrarily larger than the second term, and is bounded by* $\log(x!) < \log(x^x) = x\log x$. *This dominates the last term, so* $g(x)$ *is* $O(x\log x)$.

Question 3 (20 points) In the expressions below $\log x$ means $\log_2 x$. Show your work to get full credit.

3.1) (6 points) Find the smallest integer m such that

$$g(y) = \sqrt[3]{10^{10} - 89y^2\sqrt{y^9} + 7208y^3\sqrt[3]{y^{11}}(\log\log y)^4 - 92y^6(\log y)^2}$$

is $O(y^{m/3})$. Note that the exponent is $m/3$.

Answer: Of the terms being added under the cube root sign, $O(1)$, $O(y^{\frac{13}{2}})$, $O(y^{\frac{20}{3}}(\log\log y)^4)$, *and* $O(y^6(\log y)^2)$, *the third has the largest power of* y. *Since* $\log y = O(y^a)$ *for any* $a > 0$, *the third term is largest, and the sum is* $O(y^{\frac{20}{3}}(\log\log y)^4)$. *Taking the final cube root yields* $O(y^{\frac{20}{9}}(\log\log y)^{\frac{4}{3}})$. *This is not* $O(y^{\frac{20}{9}})$ *but is* $O(y^{\frac{21}{9}}) = O(y^{\frac{7}{3}})$ *for similar reasons, so* $m = 7$.

3.2) (7 points) The notation $\prod_{j=1}^{m} r_j$ means the product $r_1 \cdot r_2 \cdots r_m$. Find a simple, accurate function $f(m)$ such that $\prod_{j=1}^{m^3} 3^{2j+6\log j}$ is $O(f(m))$.

Answer: $\prod_{j=1}^{m^3} 3^{2j+6\log j} \leq \prod_{j=1}^{m^3} 3^{2j+6\log m^3} = \prod_{j=1}^{m^3} 3^{2j}3^{6\log m^3} = 3^{6m^3\log m^3}\prod_{j=1}^{m^3} 3^{2j} = m^{18m^3\log 3}\prod_{j=1}^{m^3} 3^{2j} = m^{18m^3\log 3}3^{\sum_{j=1}^{m^3} 2j} = m^{18m^3\log 3}3^{m^3(m^3+1)}$.

3.3) (7 points) Find a simple accurate function $r(y)$ such that
$h(y) = \log(\log((y!)^{(y!)})) + y(\log y)^{2/3}$ is $O(r(y))$.

Answer: We simplify the first term to get $\log\log(y!)^{(y!)} = \log((y!)\log(y!)) = \log(y!) + \log\log(y!)$. *The first term is arbitrarily larger than the second term, and since* $\log y! < \log y^y = y\log y$, *is* $O(y\log y)$. *This dominates the last term, so* $h(y)$ *is* $O(y\log y)$.