Math 55 Final Exam 15 Dec 2000

NAME (1 pt): ______________________

TA & section #(1 pt): ______________________

Name of Neighbor to your left (1 pt): ______________________

Name of Neighbor to your right (1 pt): ______________________

Instructions: This is a closed book, closed notes, closed calculator, closed computer, closed network, open brain exam.

You get one point each for filling in the 4 lines at the top of this page. Each part of each other question is worth 5 points. Since all parts have equal weight, do the parts you find easiest first.

You must justify your answers to get full credit.

If you start taking this exam, you have to turn it in.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn your test in (we have a stapler).

1	
2	
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Total	

Question 1) (15 points)

Note: Some of these questions ask for the smallest r such that $g(m)$ is $O(m^r)$, and some ask for the smallest r such that $g(m)$ is $O(r^m)$, so read each question carefully. Justify your answers. Your justifications may refer to results proven in class without proving them again.

1. (5 points) Find the smallest positive real number r such that $g(m) = 1/(\sum_{j=0}^{2m} 4^j)$ is $O(r^m)$.

   Answer: $g(m) = 1/((1-4^{2m+1})/(1-4)) = O((1/16)^m)$, so $r = 1/16$.

2. (5 points) Find the smallest real number r such that $g(m) = \sum_{j=1}^{m^3} j^2$ is $O(m^r)$.

   Answer: $g(m) = (m^3)^3/3 + O(m^6)$ so $g(m)$ is $O(m^9)$, so $r = 9$.

3. (5 points) Find the smallest positive real number r such that $g(m) = m^{1/2} \cdot C(m, \frac{2m}{3}) + 1.5^m$ is $O(r^m)$. You may assume that m is divisible by 3. Hint: Use Stirling's Formula, which says that $\sqrt{2\pi} m^{m+.5} e^{-m}$ is a good approximation for $m!$

   Answer: Plug in Stirling's formula into $m^{1/2} \cdot C(m, 2m/3)$ and simplify to a constant multiple of $(27/4)^{m/3}$, which is larger than $(27/8)^{m/3} = (3/2)^m$, so $r = (27/4)^{1/3}$.

Question 1) (15 points)

Note: Some of these questions ask for the smallest z such that $f(n)$ is $O(n^z)$, and some ask for the smallest z such that $f(n)$ is $O(z^n)$, so read each question carefully. Justify your answers. Your justifications may refer to results proven in class without proving them again.

1. (5 points) Find the smallest positive real number z such that $f(n) = 1/(\sum_{i=0}^{3n} 3^i)$ is $O(z^n)$.

   Answer: $f(n) = 1/((1 - 3^{3n+1})/(1 - 3)) = O((1/27)^n)$, so $z = 1/27$.

2. (5 points) Find the smallest real number z such that $f(n) = \sum_{i=1}^{n^2} i^2$ is $O(n^z)$.

   Answer: $f(n) = (n^2)^3/3 + O(n^4)$ so $f(n)$ is $O(n^6)$, so $z = 6$.

3. (5 points) Find the smallest positive real number z such that $f(n) = \sqrt{n} \cdot C(n, \frac{n}{3}) + (\frac{3}{2})^n$ is $O(z^n)$. You may assume that n is divisible by 3. Hint: Use Stirling's Formula, which says that $\sqrt{2\pi} n^{n+.5} e^{-n}$ is a good approximation for $n!$

   Answer: Plug in Stirling's formula into $\sqrt{n} \cdot C(n, n/3)$ and simplify to a constant multiple of $(27/4)^{n/3}$, which is larger than $(27/8)^{n/3} = (3/2)^n$, so $z = (27/4)^{1/3}$.

Question 1) (15 points)

Note: Some of these questions ask for the smallest s such that $h(k)$ is $O(k^s)$, and some ask for the smallest s such that $h(k)$ is $O(s^k)$, so read each question carefully. Justify your answers. Your justifications may refer to results proven in class without proving them again.

1. (5 points) Find the smallest positive real number s such that $h(k) = 1/(\sum_{r=0}^{2k} 7^r)$ is $O(s^k)$.

   Answer: $h(k) = 1/((1-7^{2k+1})/(1-7)) = O((1/49)^k)$, so $s = 1/49$.

2. (5 points) Find the smallest real number s such that $h(k) = \sum_{r=1}^{k^4} r^2$ is $O(k^s)$.

   Answer: $h(k) = (k^4)^3/3 + O(k^8)$ so $h(k)$ is $O(k^{12})$, so $s = 12$.

3. (5 points) Find the smallest positive real number s such that $h(k) = C(k, \frac{k}{3}) \cdot k^{.5} + \frac{3^k}{2^k}$ is $O(s^k)$. You may assume that k is divisible by 3. Hint: Use Stirling's Formula, which says that $\sqrt{2\pi}k^{k+.5}e^{-k}$ is a good approximation for $k!$

   Answer: Plug in Stirling's formula into $\sqrt{k}{\cdot}C(k, k/3)$ and simplify to a constant multiple of $(27/4)^{k/3}$, which is larger than $(27/8)^{k/3} = (3/2)^k$, so $k = (27/4)^{1/3}$.

Question 2) (15 points) Justify your answers.

1. (5 points) How many integers x satisfying $12345 \leq x \leq 12449$ also satisfy $x \equiv 3 \bmod 5$ and $x \equiv 6 \bmod 7$? Hint: Use the Chinese Remainder Theorem.

   Answer: There are 105 integers in the specified range. By the Chinese remainder theorem, there is a unique solution mod $5 \cdot 7 = 35$ to the two equations, i.e. a unique solution in every consecutive set of 35 integers. Since 105/35=3, there are exactly 3 integers in the specified range.

2. (5 points) Show that no integer of the form $8k + 7$ is the sum of two perfect squares $n^2 + m^2$, where k, n and m are integers. Hint: Consider the values of n^2 mod 8.

   Answer: n^2 mod 8 can equal 0, 1 or 4. So the sum of two perfect squares mod 8 can be the sum of any two numbers chosen from the set $\{0, 1, 4\}$. The possible values of this sum (mod 8) do not include 7.

3. (5 points) Let $A = \{0, 1, 2, ..., 60\}$, and consider the function $f : A \to A$ defined by $f(x) = (47x^{15} + 17) \bmod 61$. Is f one-to-one, onto, both, or neither? Hint: Use Fermat's Little Theorem.

   Answer: By Fermat's Little Theorem, $16^{15} = 2^{60} \equiv 1 \bmod 61$, so $f(1) = f(16)$, so f can neither be one-to-one nor onto.

Question 2) (15 points) Justify your answers.

1. (5 points) How many integers y satisfying $45678 \leq y \leq 45749$ also satisfy $y \equiv 2 \bmod 4$ and $y \equiv 6 \bmod 9$? Hint: Use the Chinese Remainder Theorem.

   Answer: There are 72 integers in the specified range. By the Chinese remainder theorem, there is a unique solution mod $4 \cdot 9 = 36$ to the two equations, i.e. a unique solution in every consecutive set of 36 integers. Since 72/36=2, there are exactly 2 integers in the specified range.

2. (5 points) Show that no integer of the form $8m + 3$ is the sum of two perfect squares $a^2 + b^2$, where m, a and b are integers. Hint: Consider the values of a^2 mod 8.

   Answer: $a^2 \bmod 8$ can equal 0, 1 or 4. So the sum of two perfect squares mod 8 can be the sum of any two numbers chosen from the set $\{0, 1, 4\}$. The possible values of this sum (mod 8) do not include 3.

3. (5 points) Let $B = \{0, 1, 2, ..., 46\}$, and consider the function $g : B \to B$ defined by $g(m) = (74m^{23}+23) \bmod 47$. Is g one-to-one, onto, both, or neither? Hint: Use Fermat's Little Theorem.

   Answer: By Fermat's Little Theorem, $4^{23} = 2^{46} \equiv 1 \bmod 47$, so $g(1) = g(4)$, so g can neither be one-to-one nor onto.

Question 2) (15 points) Justify your answers.

1. (5 points) How many integers z satisfying $25782 \leq z \leq 25913$ also satisfy $z \equiv 1 \bmod 3$ and $z \equiv 8 \bmod 11$? Hint: Use the Chinese Remainder Theorem.

   Answer: There are 132 integers in the specified range. By the Chinese remainder theorem, there is a unique solution mod $3 \cdot 11 = 33$ to the two equations, i.e. a unique solution in every consecutive set of 33 integers. Since 132/33=4, there are exactly 4 integers in the specified range.

2. (5 points) Show that no integer of the form $8k + 6$ is the sum of two perfect squares $r^2 + s^2$, where k, r and s are integers. Hint: Consider the values of $r^2 \bmod 8$.

   Answer: $r^2 \bmod 8$ can equal 0, 1 or 4. So the sum of two perfect squares mod 8 can be the sum of any two numbers chosen from the set $\{0, 1, 4\}$. The possible values of this sum (mod 8) do not include 6.

3. (5 points) Let $C = \{0, 1, 2, ..., 52\}$, and consider the function $h : C \to C$ defined by $h(z) = (18z^{13} + 123) \bmod 53$. Is h one-to-one, onto, both, or neither? Hint: Use Fermat's Little Theorem.

   Answer: By Fermat's Little Theorem, $16^{13} = 2^{52} \equiv 1 \bmod 53$, so $h(1) = h(16)$, so h can neither be one-to-one nor onto.

Question 3) (20 points)

Here is a game called "double or nothing" that you can play betting on coin flips. Initially your bet is $y = \$1000$. The game is played as follows:

- You flip a fair coin.
- If the coin comes up Heads, you win the amount you bet, and the game is over.
- If the coin comes up Tails, you lose the amount you bet. Then you double your bet (from y to $2y$) and play the game again.

1. (5 points) What is the probability that the game ends after exactly m coin flips? What is the total amount won or lost (adding up all the amounts won or lost on each flip) if the game ends after exactly m coin flips?

   Answer: The sample space consists of all coin flip sequences ending in the first H: $S = \{H, TH, TTH, T\cdots TH, ...\}$. The probability of $T\cdots TH$ is $.5^m$, where m is the number of coin flips. The amount lost is $1000 + 2000 + 4000 + \cdots + 1000 \cdot 2^{m-1} = 1000(2^m - 1)$ on the first $m-1$ flips, and the amount won is $1000 \cdot 2^m$ on the last flip, for a total win of 1000.

2. (5 points) What is the expected value of this game? In other words, how much do you expect to win or lose (adding up all the amounts won and lost at each coin flip)? Do you expect to win or lose money playing this game?

   Answer: The random variable f equal to the total amount won or lost in a seqence of n flips is always 1000, i.e. a constant. Therefore $E(f) = 1000$ as well. So you are guaranteed to win.

3. (5 points) Now suppose the coin is not fair, and comes up Tails with probability $1 > q > 0$. What is the value of this game? Do you expect to win more with $q = .01$ than with $q = .99$?

   Answer: The sample space is the same, but with probability function $P(T\cdots TH) = q^{m-1}(1-q)$ for a sequence of m flips. Since f is still the constant 1000, $E(f) = 1000$ independent of the probability function.

4. (5 points) Suppose the coin is fair, but you only have \$7000 to bet. This means that if you flip three Tails in a row and lose all your \$7000, the game also stops. What is the expected value of this game? What if you have \$511000 to bet instead of \$7000?

   Answer: The sample space is now $S = \{H, TH, TTH, TTT\}$, because if you throw three Tails, you lose all \$7000 and the game stops. The probability of any coin flip sequence is still $.5^m$, where m is the number of coin flips. Furthermore $f(H) = f(TH) = f(TTH) = 1000$ as before, but now $f(TTT) = -7000$. Therefore $E(f) = .5 \cdot 1000 + .5^2 \cdot 1000 + .5^3 \cdot 1000 + .5^3 \cdot (-7000) = 0$. So on average you neither win nor lose this game. If instead you have only $1000(2^m - 1)$ dollars for any m, including $1000(2^9 - 1) = 511000$, then $E(f) = \sum_{n=1}^{m} .5^n \cdot 1000 + .5^m \cdot 1000(-2^m + 1) = 0$. In other words, as long as your funds are bounded, you only expect to break even, whereas with infinite funds, you are guaranteed to win \$1000.

Question 3) (20 points)

Here is a game called "double or nothing" that you can play betting on coin tosses. Initially your bet is $x = \$10$. The game is played as follows:

- You toss a fair coin.
- If the coin comes up Tails, you win the amount you bet, and the game is over.
- If the coin comes up Heads, you lose the amount you bet. Then you double your bet (from x to $2x$) and play the game again.

1. (5 points) What is the probability that the game ends after exactly n coin tosses? What is the total amount won or lost (adding up all the amounts won or lost on each toss) if the game ends after exactly n coin tosses?

   Answer: The sample space consists of all coin toss sequences ending in the first T: $S = \{T, HT, HHT, H \cdots HT, ...\}$. The probability of $H \cdots HT$ is $.5^n$, where n is the number of coin tosses. The amount lost is $10 + 20 + 40 + \cdots + 10(2^{n-1}) = 10(2^n - 1)$ on the first $n - 1$ tosses, and the amount won is $10 \cdot 2^n$ on the last toss, for a total win of 10.

2. (5 points) What is the expected value of this game? In other words, how much do you expect to win or lose (adding up all the amounts won and lost at each coin toss)? Do you expect to win or lose money playing this game?

   Answer: The random variable f equal to the total amount won or lost in a seqence of n tosses is always 10, i.e. a constant. Therefore $E(f) = 10$ as well. So you are guaranteed to win.

3. (5 points) Now suppose the coin is not fair, and comes up Heads with probability $1 > p > 0$. What is the value of this game? Do you expect to win more with $p = .01$ than with $p = .99$?

   Answer: The sample space is the same, but with probability function $P(H \cdots HT) = p^{n-1}(1 - p)$ for a sequence of n tosses. Since f is still the constant 10, $E(f) = 10$ independent of the probability function.

4. (5 points) Suppose the coin is fair, but you only have \$70 to bet. This means that if you toss three Heads in a row and lose all your \$70, the game also stops. What is the expected value of this game? What if you have \$10230 to bet instead of \$70?

   Answer: The sample space is now $S = \{T, HT, HHT, HHH\}$, because if you throw three Heads, you lose all \$70 and the game stops. The probability of any coin toss sequence is still $.5^n$, where n is the number of coin tosses. Furthermore $f(T) = f(HT) = f(HHT) = 10$ as before, but now $f(HHH) = -70$. Therefore $E(f) = .5 \cdot 10 + .5^2 \cdot 10 + .5^3 \cdot 10 + .5^3 \cdot (-70) = 0$. So on average you neither win nor lose this game. If instead you have only $10(2^m - 1)$ dollars for any m, including $10(2^{10} - 1) = 10230$, then $E(f) = \sum_{n=1}^{m} .5^n \cdot 10 + .5^m \cdot 10(-2^m + 1) = 0$. In other words, as long as your funds are bounded, you only expect to break even, whereas with infinite funds, you are guaranteed to win \$10.

Question 3) (20 points)

Here is a game called "double or nothing" that you can play betting on die rolls. Initially your bet is $x = \$100$. The game is played as follows:

- You roll a fair die.
- If the die comes up even (2, 4 or 6) you win the amount you bet, and the game is over.
- If the die comes up odd (1, 3 or 5), you lose the amount you bet. Then you double your bet (from x to $2x$) and play the game again.

1. (5 points) What is the probability that the game ends after exactly k die rolls? What is the total amount won or lost (adding up all the amounts won or lost on each roll) if the game ends after exactly k die rolls?

   Answer: The sample space consists of all die roll sequences ending in the first E = Even: $S = \{E, OE, OOE, O \cdots OE, ...\}$, where O = Odd. The probability of $O \cdots OE$ is $.5^k$, where k is the number of die rolls. The amount lost is $100+200+400+\cdots+100(2^{k-1}) = 100(2^k - 1)$ on the first $k-1$ rolls, and the amount won is $100 \cdot 2^k$ on the last roll, for a total win of 100.

2. (5 points) What is the expected value of this game? In other words, how much do you expect to win or lose (adding up all the amounts won and lost at each die roll)? Do you expect to win or lose money playing this game?

   Answer: The random variable f equal to the total amount won or lost in a seqence of n rolls is always 100, i.e. a constant. Therefore $E(f) = 100$ as well. So you are guaranteed to win.

3. (5 points) Now suppose the die is not fair, and comes up Even with probability $1 > p > 0$. What is the value of this game? Do you expect to win more with $p = .99$ than with $p = .01$?

   Answer: The sample space is the same, but with probability function $P(O \cdots OE) = (1-p)^{k-1}p$ for a sequence of k rolls. Since f is still the constant 100, $E(f) = 100$ independent of the probability function.

4. (5 points) Suppose the die is fair, but you only have \$700 to bet. This means that if you roll three Odds in a row and lose all your \$700, the game also stops. What is the expected value of this game? What if you have \$204700 to bet instead of \$700?

   Answer: The sample space is now $S = \{E, OE, OOE, OOO\}$, because if you throw three Odds, you lose all \$700 and the game stops. The probability of any die roll sequence is still $.5^n$, where n is the number of die rolls. Furthermore $f(E) = f(OE) = f(OOE) = 100$ as before, but now $f(OOO) = -700$. Therefore $E(f) = .5 \cdot 100 + .5^2 \cdot 100 + .5^3 \cdot 100 + .5^3 \cdot (-700) = 0$. So on average you neither win nor lose this game. If instead you have only $100(2^m - 1)$ dollars for any m, including $100(2^{10} - 1) = 102300$, then $E(f) = \sum_{n=1}^{m} .5^n \cdot 100 + .5^m \cdot 100(-2^m + 1) = 0$. In other words, as long as your funds are bounded, you only expect to break even, whereas with infinite funds, you are guaranteed to win \$100.

Question 4) (10 points)

You are designing an algorithm to divide in-coming jobs between two available processors, with the goal of giving each processor about the same amount of work to do. You decide to "flip a fair coin" for each job, and send the job to Processor 1 if the coin comes up Heads, and to Processor 2 if the job comes up Tails. (In practice your algorithm would "flip a fair coin" using a random number generator, but the details don't matter here.)

1. (5 points) Suppose you do this to distribute 10000 jobs to the two processors. Show that the probability that one processor has at least 1000 more jobs than the other processor is at most 1%. Hint: Chebyshev's Inequality.

   Answer: The random variable $f = \sum_{i=1}^{10000} f_i$, where $f_i = 1$ if the i-th flip is a Head and $f_i = -1$ if the i-th flip is a Tail, says how many more (or fewer) jobs Processor 1 has than Processor 2. $E(f) = \sum_i E(f_i) = 0$ and $V(f) = \sum_i V(f_i) = \sum_i 1 = 10000$ and $\sigma(f) = 100$. So the probability that one processor has at least 1000 more jobs than the other processor is $P(|f(x)| \geq 1000) = P(|f(x) - E(f)|/\sigma(f) \geq 10) \leq 1/100$ by Chebyshev's inequality.

2. (5 points) Suppose the 10000 jobs are numbered from 1 to 10000. What is the probability that jobs 1 through 10 are assigned to the same processor and that simultaneously exactly 5000 jobs are assigned to each processor? You do not need to simplify your expression.

   Answer: $2 \cdot 2^{-10} \cdot C(9990, 5000) \cdot 2^{-9990} = C(9990, 5000) \cdot 2^{-9999}$.

Question 4) (10 points)

You are designing an algorithm to divide in-coming jobs between two available computers, with the goal of giving each computer about the same amount of work to do. You decide to "flip a fair coin" for each job, and send the job to Computer 1 if the coin comes up Tails, and to Computer 2 if the job comes up Heads. (In practice your algorithm would "flip a fair coin" using a random number generator, but the details don't matter here.)

1. (5 points) Suppose you do this to distribute 2500 jobs to the two processors. Show that the probability that one processor has at least 500 more jobs than the other processor is at most 1%. Hint: Chebyshev's Inequality.

   Answer: The random variable $f = \sum_{i=1}^{2500} f_i$, where $f_i = 1$ if the i-th flip is a Tail and $f_i = -1$ if the i-th flip is a Head, says how many more (or fewer) jobs Computer 1 has than Computer 2. $E(f) = \sum_i E(f_i) = 0$ and $V(f) = \sum_i V(f_i) = \sum_i 1 = 2500$ and $\sigma(f) = 50$. So the probability that one processor has at least 500 more jobs than the other processor is $P(|f(x)| \geq 500) = P(|f(x) - E(f)|/\sigma(f) \geq 10) \leq 1/100$ by Chebyshev's inequality.

2. (5 points) Suppose the 2500 jobs are numbered from 1 to 2500. What is the probability that jobs 1001 through 1020 are assigned to the same processor and that simultaneously exactly 1250 jobs are assigned to each processor? You do not need to simplify your expression.

   Answer: $2 \cdot 2^{-20} \cdot C(2480, 1250) \cdot 2^{-2480} = C(2408, 1250) \cdot 2^{-2499}$.

Question 4) (10 points)

You are designing an algorithm to divide in-coming jobs between two available processor units, with the goal of giving each unit about the same amount of work to do. You decide to "flip a fair coin" for each job, and send the job to Unit 1 if the coin comes up Heads, and to Unit 2 if the job comes up Tails. (In practice your algorithm would "flip a fair coin" using a random number generator, but the details don't matter here.)

1. (5 points) Suppose you do this to distribute 40000 jobs to the two units. Show that the probability that one unit has at least 4000 more jobs than the other unit is at most .0025. Hint: Chebyshev's Inequality.

   Answer: The random variable $f = \sum_{i=1}^{40000} f_i$, where $f_i = 1$ if the i-th flip is a Head and $f_i = -1$ if the i-th flip is a Tail, says how many more (or fewer) jobs Processor 1 has than Processor 2. $E(f) = \sum_i E(f_i) = 0$ and $V(f) = \sum_i V(f_i) = \sum_i 1 = 40000$ and $\sigma(f) = 200$. So the probability that one processor has at least 4000 more jobs than the other processor is $P(|f(x)| \geq 4000) = P(|f(x) - E(f)|/\sigma(f) \geq 20) \leq 1/400$ by Chebyshev's inequality.

2. (5 points) Suppose the 40000 jobs are numbered from 1 to 40000. What is the probability that jobs 111 through 160 are assigned to the same processor and that simultaneously exactly 20000 jobs are assigned to each processor? You do not need to simplify your expression.

   Answer: $2 \cdot 2^{-50} \cdot C(39950, 20000) \cdot 2^{-39950} = C(39950, 20000) \cdot 2^{-39999}$.

Question 5) (20 points)

1. (5 points) Consider the following function $c(n)$, defined for nonnegative integers n:

   - $c(0) = c(1) = 0$
   - $c(n) = c(n-1) + c(n-2) + 1$ for $n \geq 2$

   How fast does $c(n)$ grow, in a $O(\cdot)$ sense? You answer should be as simple and small as possible. Hint: What recurrence is satisfied by $d(n) = c(n) + 1$?

   Answer: $d(0) = d(1) = 1$ and $d(n) = d(n-1) + d(n-2)$, the Fibonacci recurrence. Thus $d(n) = \alpha_+ r_+^n + \alpha_- r_-^n$ for $r_\pm = (1 \pm \sqrt{5})/2$. Solving for $d(0) = d(1) = 1$ yields $\alpha_\pm = .5 \pm .5/\sqrt{5}$. So $c(n)$ is $O((\frac{1+\sqrt{5}}{2})^n)$.

2. (5 points) Now consider the function $f(n)$ defined for nonnegative integers n by the following recursive program:

   ```
   function f(n)
       if n = 0 then
           return (1)
       elseif n = 1 then
           return (2)
       else
           return(f(n − 1) * (f(n − 2))^2)
       end
   ```

   Write down a closed-form formula for $f(n)$. Hint: Consider $\log_2 f(n)$.

   Answer: Let $g(n) = \log_2 f(n)$. Then $g(0) = 0$, $g(1) = 1$, and $g(n) = g(n-1) + 2g(n-2)$ for $n \geq 2$. Now seek a solution of the form $g(n) = r^n$, and plug in to get $r^n = r^{n-1} + 2r^{n-2}$ or $r_\pm = -1$ or 2. Thus the solution is of the form $g(n) = \alpha(-1)^n + \beta 2^n$. Plugging in $g(0) = 0$ and $g(1) = 1$ yields $\alpha = -1/3$ and $\beta = 1/3$. So $g(n) = (2^n - (-1)^n)/3$, and $f(n) = 2^{g(n)}$.

3. (5 points) Using repeated squaring (and *not* logarithms and exponentials), how long (in a $O(\cdot)$ sense) does it take to compute $f(n)$ using the closed-form formula?

   Answer: First we compute $g(n)$ in $O(\log n)$ multiplies. Then we compute $2^{g(n)}$ in $\log g(n) \approx n = O(n)$ steps. This is no faster than computing $f(n)$ using the obvious loop.

4. (5 points) Let *count*(n) be the number of integer multiplications needed to compute $f(n)$ by means of the above recursive program. (Assume the that program (re)computes every value of $f(\cdot)$ needed, without storing them in a table.) Write down a recursive definition of *count*(n). How fast does *count*(n) grow, in a $O(\cdot)$ sense? Your answer should be as simple and small as possible.

   Answer: *count*$(0) = 0$, *count*$(1) = 0$, and *count*$(n) =$ *count*$(n-1) +$ *count*$(n-2) + 2$. So *count*$(n) = 2c(n)$ from the first part of the problem.

Question 5) (20 points)

1. (5 points) Consider the following function $f(m)$, defined for nonnegative integers m:

   - $f(0) = f(1) = 0$
   - $f(m) = f(m-1) + f(m-2) + 1$ for $m \geq 2$

   How fast does $f(m)$ grow, in a $O(\cdot)$ sense? You answer should be as simple and small as possible. Hint: What recurrence is satisfied by $g(m) = f(m) + 1$?

   Answer: $g(0) = g(1) = 1$ and $g(m) = g(m-1) + g(m-2)$, the Fibonacci recurrence. Thus $g(m) = \alpha_+ r_+^m + \alpha_- r_-^m$ for $r_\pm = (1 \pm \sqrt{5})/2$. Solving for $g(0) = g(1) = 1$ yields $\alpha_\pm = .5 \pm .5/\sqrt{5}$. So $g(m)$ is $O((\frac{1+\sqrt{5}}{2})^m)$.

2. (5 points) Now consider the function $h(m)$ defined for nonnegative integers m by the following recursive program:

   function $h(m)$
       if $m = 0$ then
           return (1)
       elseif $m = 1$ then
           return (2)
       else
           return$((h(m-1))^2 * (h(m-2))^3)$
       end

   Write down a closed-form formula for $h(m)$. Hint: Consider $\log_2 h(m)$.

   Answer: Let $t(m) = \log_2 h(m)$. Then $t(0) = 0$, $t(1) = 1$, and $t(m) = 2t(m-1) + 3t(m-2)$ for $m \geq 2$. Now seek a solution of the form $t(m) = r^m$, and plug in to get $r^m = 2r^{m-1} + 3t^{m-2}$ or $r_\pm = 3$ or -1. Thus the solution is of the form $t(m) = \alpha(-1)^m + \beta 3^m$. Plugging in $t(0) = 0$ and $t(1) = 1$ yields $\alpha = -1/4$ and $\beta = 1/4$. So $t(m) = (3^m - (-1)^m)/4$, and $h(m) = 2^{t(m)}$.

3. (5 points) Using repeated squaring (and *not* logarithms and exponentials), how long (in a $O(\cdot)$ sense) does it take to compute $h(m)$ using the closed-form formula?

   Answer: First we compute $t(m)$ in $O(\log m)$ multiplies. Then we compute $2^{t(m)}$ in $\log t(m) \approx m \log 3 = O(m)$ steps. This is no faster than computing $h(m)$ using the obvious loop.

4. (5 points) Let $muls(m)$ be the number of integer multiplications needed to compute $h(m)$ by means of the above recursive program. (Assume the that program (re)computes every value of $h(\cdot)$ needed, without storing them in a table.) Write down a recursive definition of $muls(m)$, How fast does $muls(m)$ grow, in a $O(\cdot)$ sense? Your answer should be as simple and small as possible.

   Answer: $muls(0) = 0$, $muls(1) = 0$, and $muls(m) = muls(m-1) + muls(m-2) + 4$. So $muls(m) = 4c(m)$ from the first part of the problem.

Question 5) (20 points)

1. (5 points) Consider the following function $h(k)$, defined for nonnegative integers k:

   - $h(0) = h(1) = 0$
   - $h(k) = h(k-1) + h(k-2) + 1$ for $k \geq 2$

   How fast does $h(k)$ grow, in a $O(\cdot)$ sense? You answer should be as simple and small as possible. Hint: What recurrence is satisfied by $t(k) = h(k) + 1$?

   Answer: $t(0) = t(1) = 1$ and $t(k) = t(k-1) + t(k-2)$, the Fibonacci recurrence. Thus $t(k) = \alpha_+ r_+^k + \alpha_- r_-^k$ for $r_\pm = (1 \pm \sqrt{5})/2$. Solving for $t(0) = t(1) = 1$ yields $\alpha_\pm = .5 \pm .5/\sqrt{5}$. So $t(k)$ is $O((\frac{1+\sqrt{5}}{2})^k)$.

2. (5 points) Now consider the function $s(k)$ defined for nonnegative integers k by the following recursive program:

   ```
   function s(k)
       if k = 0 then
           return (1)
       elseif k = 1 then
           return (2)
       else
           return(s(k − 1) ∗ (s(k − 2))^6)
       end
   ```

   Write down a closed-form formula for $s(k)$. Hint: Consider $\log_2 s(k)$.

   Answer: Let $t(k) = \log_2 s(k)$. Then $t(0) = 0$, $t(1) = 1$, and $t(k) = t(k-1) + 6t(k-2)$ for $k \geq 2$. Now seek a solution of the form $t(k) = r^k$, and plug in to get $r^k = r^{k-1} + 6t^{k-2}$ or $r_\pm = 3$ or -2. Thus the solution is of the form $t(k) = \alpha(-2)^k + \beta 3^k$. Plugging in $t(0) = 0$ and $t(1) = 1$ yields $\alpha = -1/5$ and $\beta = 1/5$. So $t(k) = (3^k - (-2)^k)/5$, and $s(k) = 2^{t(k)}$.

3. (5 points) Using repeated squaring (and *not* logarithms and exponentials), how long (in a $O(\cdot)$ sense) does it take to compute $s(k)$ using the closed-form formula?

   Answer: First we compute $t(k)$ in $O(\log k)$ multiplies. Then we compute $2^{t(k)}$ in $\log t(k) \approx k(\log 3 + \log 2) = O(k)$ steps. This is no faster than computing $s(k)$ using the obvious loop.

4. (5 points) Let $ops(k)$ be the number of integer multiplications needed to compute $s(k)$ by means of the above recursive program. (Assume the that program (re)computes every value of $s(\cdot)$ needed, without storing them in a table.) Write down a recursive definition of $ops(k)$. How fast does $ops(k)$ grow, in a $O(\cdot)$ sense? Your answer should be as simple and small as possible.

   Answer: $ops(0) = 0$, $ops(1) = 0$, and $ops(k) = ops(k-1) + ops(k-2) + 6$. [The constant 6 could be replaced by 4.] So $ops(k) = 6c(k)$ from the first part of the problem.