Discrete Mathematics - Math 55 - Spring 1997 - Final Exam

Instructions: This is a closed book, closed notes, closed calculator, closed computer, closed network, open brain exam. All 11 questions have equal weight. Write all your answers in a blue book. Put your name *and your TA's name and section number* on the blue book.

Question 1.

1. Decide if the following functions are one-to-one, onto, both, or neither. Justify your answers. N denotes the set of nonnegative integers, and Z denotes the set of integers.
   (a) Let f: $N \to Z$ be defined by $f(x) = \lfloor x \rfloor$ = "floor of x"
   (b) Let g: $N \to P(N)$, where $P(N)$ is the power set of N, be defined by $g(n) = \{1, 2, 3, ..., n\}$ for $n \geq 1$ and $g(0) = \emptyset$.
2. Let R be the set of real numbers. Let c and d be real numbers such that c is nonzero, and define the functions $h_{c,d} : R \to R$ by $h_{c,d}(x) = cx^3 + d$.
   (a) If possible, find an inverse function $k_{c,d}(x)$ for $h_{c,d}$. If it is not possible, give a one or two sentence reason why it is not possible.
   (b) Is the set of functions $\{h_{c,d} : c \in Z,\ c \neq 0,\ d \in Z\}$ countable? Give a one or two sentence justification for your answer.

Question 2.

1. Give a precise definition of what it means for $f(x)$ to be $O(g(x))$. Pay particular attention to the quantification that you use.
2. If possible, find the smallest integer k such that $f(n)$ is $O(n^k)$. Otherwise, state "not possible". Justify your answers.
   (a) $f(n) = n^{3/2} - 7$
   (b) $f(n) = \log(\log(n^n))$
   (c) $f(n) = \sin(n)$
   (d) $f(n) = 1 + 2 + 3 + \cdots + (n^3 - 1) + n^3$
   (e) $f(n) = 1 \cdot 2 \cdot 3 \cdots (n^3 - 1) \cdot n^3$
   (f) $f(n) = (2^{(6^n)})/(6^{(2^n)})$

Question 3.

1. Which integers can be represented in the form $135s + 296t$, where s and t are integers?
2. Find particular integers s and t such that $135s + 296t = 4$.
3. Solve the congruence $135x \equiv 4 \bmod 296$.

Question 4.

1. Compute $12^{198} \bmod 5$.
2. Compute $12^{198} \bmod 7$.
3. Compute $12^{198} \bmod 11$.
4. Compute $12^{198} \bmod (5 \cdot 7 \cdot 11)$.

Justify your answers. Hint: Use Fermat's Little Theorem and the Chinese Remainder Theorem.

Question 5. Find closed form formulae for the following recursively defined functions. That is, for each function find a formula for $A(n)$ that does not depend on $A(k)$ for any $k \neq n$. Justify your answers.

1. $A(0) = 1$, $A(n) = n \cdot A(n-1)$ for $n > 0$.
2. $A(0) = 1$, $A(n) = \sum_{k=0}^{n-1} (-1)^k \cdot A(k)$ for $n > 0$.
3. $A(0) = 1$, $A(n) = 1 + 2A(n-1)$ for $n > 0$.

Question 6. Prove by induction that $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$.

Question 7. A Transuranian license plate consists of a string of six digits, which must contain exactly one "3". Feel free to use expressions like the number of combinations $C(a, b)$ in your answers. Justify your answers.

1. How many plates begin with "30" or end in "3"?
2. How many plates begin with "12" or end in "12", or both?
3. How many plates contain exactly two "5"s?

Question 8. In seven-card stud poker, you get dealt seven cards from a randomly shuffled deck of 52 standard playing cards. Justify your answers. Feel free to use expressions like the number of combinations $C(a, b)$ in your answers.

1. What is the probability that you get dealt both four-of-a-kind and three-of-a-kind (e.g., four jacks and three kings) in the same hand?
2. What is the probability that you do not get at least three cards of the same suit?

Question 9.

1. A fair 52 sided die whose sides are labeled with the names of the standard playing cards is rolled until the ace of spades is rolled. Let r be the total number of rolls. What is the expected value $E(r)$? What is the variance $V(r)$? Justify your answers.
2. From a shuffled deck of standard playing cards, cards are drawn and discarded until the ace of spades is drawn. What is the expected number of draws? Is the expected number of draws less than, equal to, or greater than the expected number of rolls in the first part of this question? Justify your answers.

Question 10. Let s be the sum of the values from tossing a fair six-sided die 101 times, with its sides marked 1,2,3,4,5, and 6. What is the probability that s is even? Justify your answer.

Question 11. Let n be a positive integer. Prove that at least 75% of all subsets of a set with n^2 elements have cardinalities lying between $n(n-2)/2$ and $n(n+2)/2$ (exclusive). Hint: Chebyshev's Inequality.