

Math 55  
Review for Final Exam

**“Coins”**

1. A biased coin has the probability of heads four times as large as the probability of tails.
  - (a) What is the probability that when the coin is tossed seven times, the results of the flips alternate between heads and tails?
  - (b) What is the expected number of tails in seven flips?
  - (c) How many times must the coin be flipped so that there is at least a 90% chance that there has been at least one tail?
  - (d) Suppose the coin is flipped 100 times. Give a lower bound on the probability that the number of tails is between thirteen and twenty-seven, inclusive.
2. Find the probability that when a fair coin is flipped five times, tails comes up exactly three times, the first and last flips are tails, or the second and fourth flips are tails.
3. Suppose that in independent Bernoulli trials, the probability of success is  $p$ .
  - (a) What is the probability that the tenth success occurs on the thirteenth trial?
  - (b) Let  $X$  denote the number of successes that occur in the first ten trials. Let  $Y$  denote the number of failures that occur among the outcomes of the eleventh and twelfth trials. What are the variances of  $X$ ,  $Y$ , and  $X + Y$ ?
4. You are given a trick coin for which the probability of heads on the  $i$ th flip is  $\frac{1}{i}$ . What is the expected number of heads in 100 flips? The variance?
5. A fair nickel and a fair penny are tossed simultaneously until both come up tails at the same time. What is the expected number of times the coins are tossed?
6. What is the expected number of boys in a family of six children if the sexes of the children are independent and the probability that the  $i$ th child is a boy is  $\frac{1}{2} - \frac{i}{20}$ ? What's the variance?

**Dice**

7. We are playing with fair fifty-sided dice numbered 1–50. Suppose we roll three of them.
  - (a) What is the expected sum of the numbers on the dice?
  - (b) What is the probability that the sum of the numbers on the dice is 37?
  - (c) What is the probability that the sum of the numbers on the dice is 53?
  - (d) Repeat (a)–(c) if it is known that the number on the first die is 35.
  - (e) Repeat (a)–(c) if it is known that the number 35 was rolled on one of the dice.
8. A four-sided die is weighted so that  $P(1) = .3$ ,  $P(2) = .2$ ,  $P(3) = .4$ . If you roll five dice like this, what is the expected sum? What is the probability of getting exactly three 2's?
9. Twenty fair six-sided dice are rolled. Show that the probability that the sum is greater than or equal to 100 is less than 4%.
10. I roll a single die repeatedly until three different numbers have come up. What is the expected number of times that I need to roll the die?
11. I roll three fair dice numbered 2–7. What is the probability that their sum is thirteen?
12. What is the expected sum of the numbers that appear on three dice, each biased so a 4 comes up twice as often as each other number?

13. A pair of dice are rolled until either a sum of 5 or 7 comes up. Find the probability that the 5 comes first.
14. Let  $s$  be the sum of the values from rolling a fair six-sided die 101 times. What is the probability that  $s$  is even?

## Cards

15. (a) In a shuffled deck, cards are drawn and discarded until the ace of spades is drawn. What is the expected number of draws?  
 (b) Now we roll a 52-sided die (with the faces labeled with the standard playing cards) until the ace of spades shows up. What is the expected number of rolls? Compare to part (a).
16. What is the probability that a five-card poker hand contains no aces, exactly one king, and at least one heart?
17. What is the probability that a seven-card poker hand contains
  - (a) four cards of one kind and three cards of a second kind?
  - (b) three cards of one kind and pairs of each of two different kinds?
  - (c) pairs of each of three different kinds and a single card of a fourth kind?
  - (d) pairs of each of two different kinds and one card each of three different kinds?
  - (e) cards of seven different kinds?
  - (f) a seven-card straight?
  - (g) a seven-card flush?
  - (h) a seven-card straight flush?
  - (i) no more than two cards in any one suit?
18. Compute the probability that a bridge hand (thirteen cards) has a “void;” *i.e.*, is missing at least one suit.
19. In bridge, there is a saying “Eight ever, nine never.” This is a mnemonic device that tells you how to play a certain situation that comes up fairly often. Without going into any detail, here is what you need to know to solve this problem:

In some situations, you have to decide whether to try to “finesse” or try to “drop” the queen of a certain suit. The finesse “works” whenever the queen is singleton (*i.e.*, it’s the only card of that suit an opponent holds); or if the opponent sitting to your right has it and at most three other cards in the suit. The drop “works” whenever the queen is at worst doubleton (*i.e.*, the opponent with the queen has at most one other card in that suit), regardless of which opponent has it; or when your right hand opponent has all of the missing cards in the suit, as long as he has four or fewer cards in that suit. There are four players, each of whom has thirteen cards, and you know both your hand and your partner’s hand. Show that if you and your partner together have eight cards in a suit, then your odds are better to try the finesse, while if you and your partner collectively have nine cards in the suit, the drop is the better play.

## Miscellaneous

20. Each box of the cereal “Flakey” contains a small plastic letter. Only the letters A, E, F, K, L, Y occur, and with equal probabilities.
  - (a) What is the expected number of boxes one must buy until one has two distinct letters?
  - (b) What is the expected number of boxes one must buy until one has all six letters?

21. If eight rooks are randomly placed on a chessboard, compute the probability that none of them can capture any other. In other words, what is the probability that no two are on the same row or file?
22. Two balls are painted either blue or gold. Suppose each is painted blue with probability  $\frac{1}{2}$  and that these events are independent. The balls are then placed into an urn.
- You notice that some gold paint has been used, so at least one ball is gold. Compute the probability that both balls are painted gold.
  - You start the experiment over and this time the person doing it is careful to not let you notice what kind of paint was used. Unfortunately, the urn tips over and one of the balls spills out. It is painted gold. What is the probability that both balls are gold?
23. Let  $S$  be the sample space of two-digit odd integers (with leading zeros allowed)  $n = d_1d_2$  with  $01 \leq n \leq 99$  chosen at random with equal probabilities.
- Find the probability of the event  $F$  that an element of  $S$  has distinct digits  $d_1 \neq d_2$ .
  - Define a random variable  $f$  to be the sum of the digits of  $n$ , so  $f(n) = d_1 + d_2$ . Find the expectation  $E(f)$ .
  - Let  $G$  be the event that  $f(n) = 10$ . Find the conditional probability  $P(G|F)$ .
  - Suppose we choose an integer  $n$  at random from the new sample space  $F$  instead of  $S$ , with equal probabilities. What is the expected sum of the digits of  $n$ ?
24. Choose a nonnegative integer solution of the equation  $x_1 + x_2 + x_3 = 42$  at random, with equal probabilities. What is the probability that at least one of the  $x_i$ 's is exactly equal to 7?
25. Suppose that the number of tin cans recycled in a day at a recycling center is a random variable with an expected value of 50,000 and a variance of 2,500. Provide a lower bound on the probability that the center will recycle between 40,000 and 60,000 cans on a certain day.
26. A box contains five red and three black marbles. Two marbles are drawn at random without replacement. What is the probability that they are the same color?
27. Suppose  $P(A) = \frac{1}{3}$ ,  $P(B) = \frac{1}{2}$ ,  $P(A \cup B) = \frac{2}{3}$ .
- What is  $P(A \cup B|A)$ ?
  - What is  $P(A \cap B|A \cup B)$ ?
28. If  $E$  and  $F$  are not independent, then  $P(E \cap F) \neq P(E) \cdot P(F)$ . Show that both  $<$  and  $>$  are possible by giving examples.
29. Let  $n$  be a positive integer. Prove that at least 75% of all subsets of a set with  $n^2$  elements have cardinalities lying strictly between  $n(n-2)/2$  and  $n(n+2)/2$ .

For review question from previous subjects, see the first two review sheets.

Other good sources of problems are the textbook and Lenstra's notes. Since the answers to all of the odd-numbered exercises in the textbook are given in the back and Professor Demmel only assigns even-numbered problems, you should have plenty of material to work with. Try doing problems that are similar to ones on the homework.

## Answer Key

1. (a)  $\frac{4^3+4^4}{5^7} = \frac{320}{78,125}$   
 (b)  $\frac{7}{5}$   
 (c) 11  
 (d)  $\frac{3}{4}$
2.  $\frac{9}{16}$
3. (a)  $(\frac{12}{9})p^{10}(1-p)^3$   
 (b)  $V(X) = 10p(1-p)$   
 $V(Y) = 2p(1-p)$   
 $V(X+Y) = 12p(1-p)$
4.  $E(f) = \sum_{i=1}^{100} \frac{1}{i} \approx 5.19$   
 $V(f) = \sum_{i=1}^{100} \frac{1}{i^2} (1 - \frac{1}{i}) \approx 3.55$
5. 4
6.  $E(b) = \frac{39}{20}$   
 $V(b) = \frac{509}{400}$
7. (a)  $\frac{153}{2}$   
 (b)  $\frac{\binom{36}{2}}{50^2} = \frac{630}{125,000}$   
 (c)  $\frac{\binom{52}{2}-3}{50^2} = \frac{1323}{125,000}$   
 (d)  $E(s) = 86$   
 $P(s = 37) = \frac{1}{2,500}$   
 $P(s = 53) = \frac{17}{2,500}$   
 (e)  $E(s) = \frac{1,267,203}{14,702} \approx 86.19$   
 $P(s = 37) = \frac{3}{7,351}$   
 $P(s = 53) = \frac{51}{7,351}$
8.  $E(s) = 11.5$   
 $P(\text{exactly 3 2's}) = \binom{5}{3} (\frac{1}{5})^3 (\frac{4}{5})^2 = \frac{32}{625}$
10.  $1 + \frac{6}{5} + \frac{6}{4} = \frac{37}{10}$
11.  $\frac{\binom{9}{2} - \binom{3}{2} \cdot 3}{6^3} = \frac{1}{8}$
12.  $\frac{75}{7}$
13.  $\sum_{i=0}^{\infty} (\frac{13}{18})^i \cdot \frac{1}{9} = \frac{2}{5}$
14.  $\frac{1}{2}$
15. (a)  $\frac{53}{2}$   
 (b) 52
16.  $\frac{\binom{44}{4} + \binom{3}{1} [\binom{44}{4} - \binom{33}{4}]}{\binom{52}{5}} \approx 16.2\%$
17. All of the answers have a denominator of  $\binom{52}{7}$ ; the given numbers are the numerators.  
 (a)  $\binom{13}{1} \binom{12}{1} \binom{4}{4} \binom{4}{3} = 624$   
 (b)  $\binom{13}{1} \binom{12}{2} \binom{4}{3} \binom{3}{2}^2 = 123,552$   
 (c)  $\binom{13}{3} \binom{10}{1} \binom{4}{2}^3 \binom{4}{1} = 2,471,040$   
 (d)  $\binom{13}{2} \binom{11}{3} \binom{4}{2}^2 \binom{4}{1}^3 = 29,652,480$   
 (e)  $\binom{13}{7} \binom{4}{1}^7 = 28,114,944$   
 (f)  $\binom{8}{1} \binom{4}{1}^7 = 131,072$   
 (g)  $\binom{4}{1} \binom{13}{7} = 65,536$   
 (h)  $\binom{8}{1} \binom{4}{1} = 32$   
 (i)  $\binom{4}{3} \binom{13}{2}^3 \binom{13}{1} = 24,676,704$
18.  $\frac{\binom{4}{1} \binom{39}{13} - \binom{4}{2} \binom{26}{13} + 4}{\binom{52}{13}} \approx 5.1\%$
19. When you have eight cards, the finesse works approximately 50.9% of the time and the drop works approximately 32.8% of the time.  
 When you have nine cards, the finesse works approximately 56.2% of the time and the drop works approximately 57.9% of the time.
20. (a)  $\frac{11}{5}$   
 (b)  $\frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = \frac{147}{10}$
21.  $\frac{8!}{\binom{64}{8}}$
22. (a)  $\frac{1}{3}$   
 (b)  $\frac{1}{2}$
23. (a)  $\frac{9}{10}$   
 (b)  $\frac{19}{2}$   
 (c)  $\frac{4}{45}$   
 (d)  $\frac{85}{9}$
24.  $\frac{105}{946}$
25. .999975
26.  $\frac{13}{28}$
27. (a) 1  
 (b)  $\frac{1}{4}$