Welcome to Ma221! (Apr 14)

Continue with Splitting Methods

$$A = M - K$$

(*) solve $Mx_{i+1} = Kx_i + b$ for x_{i+1}

$$x_{i+1} = M^{-1}Kx_i + M^{-1}b = Rx_i + c$$

Thm: (*) converges to $A^{-1}b$ $\forall x_0$ iff $\rho(R) = \max_i |\lambda| < 1$

$$A = D - L' - U' = D(I - L - U)$$

Jacobi: $Dx_{i+1} = (L' + U')x_i + b$

$$R_J = L + U$$

GS $(D - L')x_{i+1} = U'x_i + b$

$$R_{GS} = (I - L)^{-1}U$$

SOR(w): $\left(\frac{1}{w}D - L'\right)x_{i+1} = \left(\left(\frac{1}{w} - 1\right)D + U'\right)x_i + b$

$$R_{SOR(w)} = (I - wL)^{-1}\left((1-w)I + wU\right)$$

$w = 1 \Rightarrow$ same as GS

Thm: If A strictly row diagonally dominant

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}| \quad \forall i$$

then J and GS both converge,

$$\|R_{GS}\|_\infty \le \|R_J\|_\infty < 1$$

GS at least as fast as J.

Doesn't apply to Poisson:

$$-1 \quad -1 \;\; 4 \;\; -1 \quad -1$$

Def: A weakly row diagonally dominant if $|A_{ii}| \ge \sum_{j \ne i} |A_{ij}| \;\; \forall i$

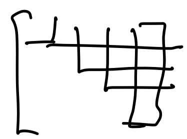
and strict ($>$) at least once

True for Poisson: $-1 \;\; 4 \;\; -1 \;\; -1$ for some rows

Not enough for convergence:

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \Rightarrow R^5 = R$$

Def: A matrix is irreducible if there is no permutation P such that

$PAP^T =$  is block triangular

Equivalently: the directed graph corresponding to A ($A(i,j) \ne 0 \Leftrightarrow$ edge (i,j)) is strongly connected:

i.e. path from all i to all j

Model Problem is irreducible because graph is a 1D, 2D, 3D, ... mesh

Model Problem also weakly row diag. dom.

Thm: If A weakly row diag dom and irreducible then $\rho(R_{GS}) < \rho(R_J) < 1$

so GS and J both converge, GS faster [Thm 6.3 in text]

Thm: A s.p.d $\Rightarrow$ then SOR(ω) converges iff $0 < \omega < 2$

In particular SOR(1) = GS

[Thm 6.5 in text]

Convergence of SOR(ω) for 2D Poisson

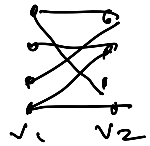
One more graph theortic property:

Def: A matrix has "Property A" if there is a permutation such that

$$P A P^T = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad A_{11}, A_{22} \text{ diagonal}$$

Graph Theoretically: A is "bipartite"

: Partition nodes (rows + cols) $V = V_1 \cup V_2$ such that all edges in graph go from V_1 to V_2, V_2 to V_1 none from V_1 to V_1, V_2 to V_2



For 1D Poisson

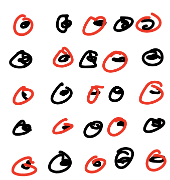
odd vertices $= V_1$

even " $= V_2$

For 2D Poisson

$j+k$ odd $= V_1$

$j+k$ even $= V_2$



Same idea for 3D Poisson etc

Thm: Suppose A has "Property A" and we do SOR(ω) updating vertices in V_1 before V_2. Then evals μ of R_J and evals λ of $R_{SOR(\omega)}$ are related by

$$(*)\quad (\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2$$

If $\omega = 1$, so SOR(1) = GS then $\lambda = \mu^2$

$\Rightarrow \rho(R_{SOR(1)}) = \rho(R_{GS}) = (\rho(R_J))^2$

$\Rightarrow$ GS converges twice as fast as J

(see Thm 6.6 in text)

proof: number vertices in V_1 before V_2

$$\Rightarrow A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{22} & A_{22} \end{array}\right] \quad A_{ii} \text{ diagonal}$$

for 2D Poisson $A = 4I + \left[\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & 0 \end{array}\right]$

$$= 4I + \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}$$

$$= D - L' - U'$$

λ eval of $R_{SOR(\omega)} \Rightarrow$

$$0 = \det(\lambda I - R_{SOR(\omega)})$$

$$= \det(\lambda I - (I - \omega L)^{-1}((1-\omega)I + \omega U))$$

$$= \det((I - \omega L)(\ \text{"}\ ))$$

$$= \det(\lambda I - \omega \lambda L - (1-\omega)I - \omega U)$$

$$= \det((\lambda - 1 + \omega)I - \omega \lambda L - \omega U)$$

$$= \det\left(\sqrt{\lambda}\omega\left(\frac{\lambda - 1 + \omega}{\sqrt{\lambda}\omega} I - \sqrt{\lambda} L - \frac{1}{\sqrt{\lambda}} U\right)\right)$$

$$= (\sqrt{\lambda}\omega)^n \det\left(\frac{\lambda - 1 + \omega}{\sqrt{\lambda}\omega} I - \sqrt{\lambda} L - \frac{1}{\sqrt{\lambda}} U\right)$$

$$D = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} I \end{bmatrix}$$

$$D(\sqrt{\lambda} L) D^{-1} = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{\lambda} A_{21} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \sqrt{\lambda} I \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} = L$$

$$D\left(\frac{1}{\sqrt{\lambda}}U\right)D^{-1} = U$$

$$= c\cdot\det\left(D\,(\text{''})\,D^{-1}\right)$$

$$= c\cdot\det\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega}I \underbrace{-L-U}_{-R_J}\right)$$

$$= c\cdot\det\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega}I - R_J\right)$$

$$\Rightarrow \frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} = \text{eval of } R_J = \mu$$

$\dfrac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} = \mu$, square, multiply by denom, get (*)

Since we know all μ of R_J for Poisson

can compute all λ of $R_{SOR(\omega)}$

and pick ω to minimize $\rho(R_{SOR(\omega)})$

Thm: Suppose A has "property A"

and SOR(ω) updates V_1 before V_2

and $\mu = \rho(R_J) < 1$ so Jacobi converges

then $\omega_{OPT} = \dfrac{2}{1+\sqrt{1-\mu^2}}$

$$\rho(R_{SOR(\omega_{OPT})}) = \omega_{OPT} - 1 = \frac{\mu^2}{\left(1+\sqrt{1-\mu^2}\right)^2}$$

for 2D Poisson: $\omega_{OPT} = \dfrac{2}{1+\sin\left(\frac{\pi}{n+1}\right)} \lesssim 2$

$$\rho\left(R_{SOR(\omega_{opt})}\right) = \frac{\cos^2\left(\frac{\pi}{n+1}\right)}{\left(1+\sin\left(\frac{\pi}{n+1}\right)\right)^2} \approx 1 - \frac{2\pi}{n+1}$$

if n large

$\Longrightarrow$ # steps to converge $\sim \sqrt{\text{\# steps for Jacobi or GS}}$