Welcome to Ma 221! (Mar 10)

Eigenvalue Problems

Goals:

- Canonical Forms (recall Jordan, why Schur form is better)
- Variations on eigenproblems (not just one square matrix)
- Perturbation theory (can I trust my answer?)
- Algorithms (for a single nonsymmetric matrix: Chap 5 for $A=A^T$)

Webpage: Templates for Solution of Algebraic Eigenvalue Problems

Recall Definitions

Def: $p(\lambda) = \det(A^{n \times n} - \lambda I)$ is characteristic polynomial

n roots $\rightarrow$ eigenvalues

Def: if λ eigenvalue, $\exists$ nonzero null vector x of $(A-\lambda I)x \Rightarrow Ax=\lambda x$

x right eigenvector

Analogously $\exists\, y^H$ s.t. $y^H(A-\lambda I)=0$

$\Rightarrow y^H A = \lambda y^H$, y^H left eigenvector

Def: S nonsingular and $B = SAS^{-1}$

S is a similarity transform

A and B similar

Lemma: A and B similar $\Rightarrow$ have same evals, and evecs related by multiplying by S:

pf: $Ax=\lambda x$ iff $\underbrace{SAS^{-1}}_{B}Sx = \lambda Sx$

$$B(Sx) = \lambda(Sx)$$

$y^H A = \lambda y^H$ iff $y^H S^{-1}\underbrace{SAS^{-1}}_{B} = \lambda y^H S^{-1}$

$$(y^H S^{-1})B = \lambda(y^H S^{-1})$$

Goal: Transform A to a simpler and similar B whose evals and evecs are "easy" to compute

Simplest: B diagonal

$\Rightarrow$ eigenvalues are $B(i,i)$, evecs $= \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} i$

Lemma: if $Ax_i = \lambda_i x_i$ for $i=1:n$

and $S = [x_1, \dots, x_n]$ nonsingular

i.e. $\exists$ n linearly independent evecs

Then $A = S \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \cdot S^{-1}$

Conversely if $A = S \Lambda S^{-1}$, Λ diagonal

then columns of S are evecs

proof: $A = S\Lambda S^{-1}$ iff $AS = S\Lambda$

iff $AS[:,i] = \underbrace{S[:,i]}_{\text{evec}} \cdot \underbrace{\lambda_i}_{\text{eval}} \quad \forall i$

But we can't always "diagonalize" A

for 2 reasons:

may be mathematically impossible

(recall Jordan Form)

may be unstable, even if it exists

(when eigenvalues very close)

Recall Jordan Form: For any A

$\exists$ similarity $SAS^{-1} = J = \mathrm{diag}(J_1, \ldots, J_k)$

where each $J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$

Up to permuting order of J_i,

this is unique. different J_i can

have equal λ_i, eg $A = I$

Only one right or left evec per block

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

can't diagonalize

Why not compute Jordan Form?

Consider slightly perturbed 2×2 I

(1) $\begin{bmatrix} 1 & 0 \\ 0 & 1+e \end{bmatrix} : \left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(1+e, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

(2) $\begin{bmatrix} 1 & e \\ e & 1 \end{bmatrix} : \left(1+e, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \left(1-e, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

evecs rotated $45°$

(3) $\begin{bmatrix} 1 & e \\ 0 & 1+e^2 \end{bmatrix}$: $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(1+e^2, \begin{bmatrix} 1 \\ e \end{bmatrix}\right)$

evecs nearly parallel

(4) $\begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix}$: $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ just one evec

(5) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: $(1, \text{anything}), (1, \text{anything})$

When evals (nearly) identical
Jordan Form very ill conditioned

Our goal: backward stability:
exact evals, evecs of $A+E$, $\frac{\|E\|}{\|A\|} = O(\varepsilon)$

Backward Stable approach to eigen problem:
Recall Chap 3: Multiplying by multiple orthogonal matrices backward stable

$$fl(Q_k(\cdots(Q_2(Q_1 A))\,)\,) = \underset{\text{exact orthogonal}}{Q}(A + \underset{\frac{\|E\|}{\|A\|}=O(\varepsilon)}{E})$$

Apply twice to orthogonal similarity

$$fl(Q_k(\cdots(Q_2(Q_1 A Q_1^T)Q_2^T)\cdots)Q_k^T) = \underset{\text{exact orthog}}{Q}(A + \underset{\frac{\|F\|}{\|A\|}=O(\varepsilon)}{F})Q^T$$

If we restrict similarity S to be orthogonal, how close to Jordan form can we get?

Thm (Schur canonical form)

Given an $n \times n$ A $\exists$ unitary Q $QQ^H = I$ s.t. $Q^H A Q = T$ = upper triangular with evals $T(i,i)$ which can appear in any order

Computing evecs of T: just triangular solve

$$\begin{array}{c} \\ i-1 \\ 1 \\ n-i \end{array}
\begin{array}{c} \begin{array}{ccc} i-1 & 1 & n-i \end{array} \\ \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ & T(i,i) & T_{23} \\ & & T_{33} \end{bmatrix} \end{array}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\begin{array}{c} i-1 \\ 1 \\ n-i \end{array}
= T(i,i) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T_{11}x_1 + T_{12}x_2 + T_{13}x_3 = T(i,i)x_1$$
$$T(i,i)x_2 + T_{23}x_3 = T(i,i)x_2$$
$$T_{33}x_3 = T(i,i)x_3$$

if $T(i,i)$ unique then only solution to $(T_{33} - T(i,i)I)x_3 = 0$ is $x_3 = 0$

$$T(i,i)x_2 = T(i,i)x_2 : \quad x_2 = 1$$

$$(T_{11} - T(i,i)I)x_1 = -T_{12}$$

nonsingular triangular system, since $T(i,i)$ unique

What does $[V,D] = \text{eig}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ do? try it!

proof that Schur Form exists:

choose any λ, corresponding x: $Ax = \lambda x$, $\|x\|_2 = 1$

Let $Q = [x, Q']$ be unitary

$$Q^H A Q = \begin{bmatrix} x^H \\ Q'^H \end{bmatrix} A \begin{bmatrix} x, & Q' \end{bmatrix}$$

$$= \begin{array}{c} \\ 1 \\ n-1 \end{array} \begin{array}{c} \begin{array}{cc} 1 & n-1 \end{array} \\ \left[\begin{array}{c|c} x^H A x & x^H A Q' \\ \hline Q'^H A x & Q'^H A Q' \end{array}\right] \end{array}$$

$$= \left[\begin{array}{c|c} \lambda x^H x & x^H A Q' \\ \hline \lambda Q'^H x & Q'^H A Q' \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda & x^H A Q' \\ \hline 0 & Q'^H A Q' \end{array}\right]$$

apply induction to $Q'^H A Q' = U^H \underset{\nabla}{T} \overset{\text{unitary}}{U}$

$$Q^H A Q = \left[\begin{array}{c|c} \lambda & x^H A Q' \\ \hline 0 & U^H T U \end{array}\right]$$

$$= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U^H \end{array}\right] \left[\begin{array}{c|c} \lambda & x^H A Q' U^H \\ \hline 0 & T \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U \end{array}\right]$$

$$\Rightarrow \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U \end{array}\right] Q^H}_{\text{unitary}} A \underbrace{Q \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U^H \end{array}\right]}_{\substack{\text{unitary} \\ \text{(inverse)}}} = \left[\begin{array}{c|c} \lambda & x^H A Q' U^H \\ \hline 0 & T \end{array}\right]$$

what about real matrices
with complex evals?

Real matrices can have complex eigenvalues, special case: $A = A^T$
all real evals, Chap 5

Prefer real arithmetic for real A:

reduce # flops

less memory

make sure that evals, evecs appear in complex conjugate pairs

$$\underset{\text{real}}{A}\,\underset{\text{complex}}{x = \lambda x} \Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$$

Instead of T = upper triangular use block triangular

$$T = \begin{bmatrix} T_{11} & & T_{ij} \\ & T_{22} & \\ 0 & & \ddots & T_{kk} \end{bmatrix}$$

evals of $T = \bigcup_i$ evals of T_{ii} (HW 4.1)

goal: have 2×2 blocks T_{ii} for complex conjugate evals

each T_{ii} either real and 1×1 or 2×2 with eigs $\lambda, \bar{\lambda}$

Thm (Real Schur Canonical Form):

Given real A, $\exists$ real orthogonal Q s.t. QAQ^T is block upper triangular with 1×1 and 2×2 blocks

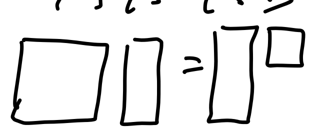
Generalize to "invariant subspace"

Def: $V = \text{span}\{x_1, \dots, x_m\} = \text{span}(X)$ $\quad X = [x_1, \dots, x_m]$
be a subspace of $\mathbb{R}^n$. V invariant
if $A \cdot V = \text{span}(A \cdot X) \subseteq V$

Ex: $V = \text{span}\{x\} = \{\alpha x \text{ for all scalars } \alpha\}$
where $Ax = \lambda x$
$AV = \{A(\alpha x), \text{ all } \alpha\} = \{\alpha \lambda x, \text{ all } \alpha\}$
$\subseteq V$ $(= V \text{ unless } \lambda = 0)$

Ex: $V = \text{span}\{x_1, \dots, x_k\} = \{\sum_{i=1}^{k} \alpha_i x_i \;\; \forall \alpha_i\}$
where $Ax_i = \lambda_i x_i$
$AV = \{A(\sum_i \alpha_i x_i)\} = \{\sum_i \alpha_i A x_i\} = \{\sum_i \alpha_i \lambda_i x_i\}$
$\subseteq V$

Lemma: $V = \text{span}\{x_1, \dots, x_m\} = \text{span}(X)$
be invariant subspace of A
Then $\exists B$: $AX = X \cdot B$



the evals of B are evals of A

proof: existence of B follows from def:

$Ax_i \in V \Rightarrow \exists$ scalars $B(1,i), B(2,i) \dots B(m,i)$

s.t. $Ax_i = \sum_{j=1}^{m} x_j B(j,i)$ i.e. $AX = XB$

$$By = \lambda y \Rightarrow A(Xy) = XBy = X\lambda y = \lambda(Xy)$$

Lemma: $V = \text{span}(X)$ be m-dimensional invariant subspace, so $AX = XB$

$X = QR$ Let $[\overset{m}{Q}, \overset{n-m}{Q'}]$ be square, orthogonal

$$[Q, Q']^T A [Q, Q'] = \begin{array}{cc} & \begin{array}{cc} m & n-m \end{array} \\ \begin{array}{c} m \\ n-m \end{array} & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] \end{array}$$

$A_{11} = RBR^{-1}$ has same evals as B

proof: $[Q, Q']^T A [Q, Q'] = \begin{bmatrix} Q^T A Q & Q^T A Q' \\ Q'^T A Q & Q'^T A Q' \end{bmatrix}$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$AQ = AXR^{-1} = XBR^{-1} = QRBR^{-1}$$

$$A_{11} = Q^T A Q = \underbrace{Q^T Q}_{I} RBR^{-1} = RBR^{-1}$$

$$A_{21} = \underbrace{Q'^T Q}_{0} RBR^{-1} = 0$$

Proof of Real Schur form:

if $Ax=\lambda x$, x,λ real reduce to $n-1 \times n-1$ problem as before

if x,λ complex, take real, imag parts of $Ax=\lambda x$

$$X=[re(x), im(x)] \quad B=\begin{bmatrix} re(\lambda) & im(\lambda) \\ -im(\lambda) & re(\lambda) \end{bmatrix}$$

$AX=XB$ first col $real(Ax)=real(\lambda x)$

second " $imag(Ax)=imag(\lambda x)$

X invariant subspace, real

evals(B) are $\lambda, \bar{\lambda}$

use Lemma to do orthogonal (real) similarity to get

$$\begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \end{array}$$

evals of $A_{11}=RBR^{-1}$ are $\lambda, \bar{\lambda}$

Review other eigenproblems

(1) ODE $x'(t)=Kx(t)$

If $Kx(0)=\lambda x(0)$ then

$x(t)=e^{\lambda t}x(0)$

similar if $x(0)=$ linear combo of evecs

(2) $$Mx''(t) + Kx(t) = 0$$

$$\Rightarrow \lambda^2 M x(0) + K x(0) = 0$$

"generalized eigen problem"

$$\det(\lambda' M + K) = 0 \quad \text{where } \lambda' = \lambda^2$$

(3) $$Mx''(t) + D x'(t) + K x(t) = 0$$

$\Rightarrow$ "nonlinear eigenproblems"

$$\lambda^2 M x(0) + \lambda D x(0) + K x(0) = 0$$

reduce to linear problem (2 matrices) of 2x size

(4) $$x'(t) = A x(t) + B u(t)$$

"linear control system"

turns into singular eigenproblem

$$\begin{array}{cc} m & n \\ \left[B, \right. & \left. A \right] \end{array}^{n} \text{ and } \begin{array}{cc} m & n \\ \left[0, \right. & \left. I \right] \end{array}^{m}$$

All ideas of Chap 4:

Jordan form, Schur form, algs

generalize to all these cases

We concentrate on one

nonsymmetric A

(see Chap 4.5 for other cases)

Next topic: Perturbation Theory