
Multigrid


James Demmel

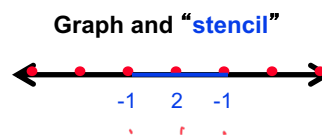
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Poisson's equation in 1D: $\partial^2 u / \partial x^2 = f(x)$

$$T = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$




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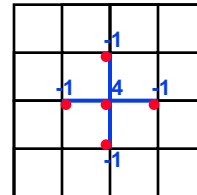
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2D Poisson's equation

- Similar to the 1D case, but the matrix T is now

$$T = \begin{pmatrix} 4 & -1 & & -1 & & \\ -1 & 4 & -1 & & -1 & \\ & -1 & 4 & & & -1 \\ -1 & & & 4 & -1 & -1 \\ & -1 & -1 & 4 & -1 & \\ & & -1 & -1 & 4 & -1 \\ & & & -1 & & 4 & -1 \\ & & & & -1 & & 4 & -1 \\ & & & & & -1 & & 4 \end{pmatrix}$$

Graph and "stencil"



- 3D is analogous

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Algorithms for 2D (3D) Poisson Equation ($N = n^2$ (n^3) vars)

Algorithm	Serial	PRAM	Memory	#Procs
Dense LU	N^3	N	N^2	N^2
Band LU	N^2 ($N^{7/3}$)	N	$N^{3/2}$ ($N^{5/3}$)	N ($N^{4/3}$)
Jacobi	N^2 ($N^{5/3}$)	N ($N^{2/3}$)	N	N
Explicit Inv.	N^2	$\log N$	N^2	N^2
Conj.Gradients	$N^{3/2}$ ($N^{4/3}$)	$N^{1/2(1/3)} * \log N$	N	N
Red/Black SOR	$N^{3/2}$ ($N^{4/3}$)	$N^{1/2}$ ($N^{1/3}$)	N	N
Sparse LU	$N^{3/2}$ (N^2)	$N^{1/2}$	$N * \log N$ ($N^{4/3}$)	N
FFT	$N * \log N$	$\log N$	N	N
Multigrid	N	$\log^2 N$	N	N
Lower bound	N	$\log N$	N	

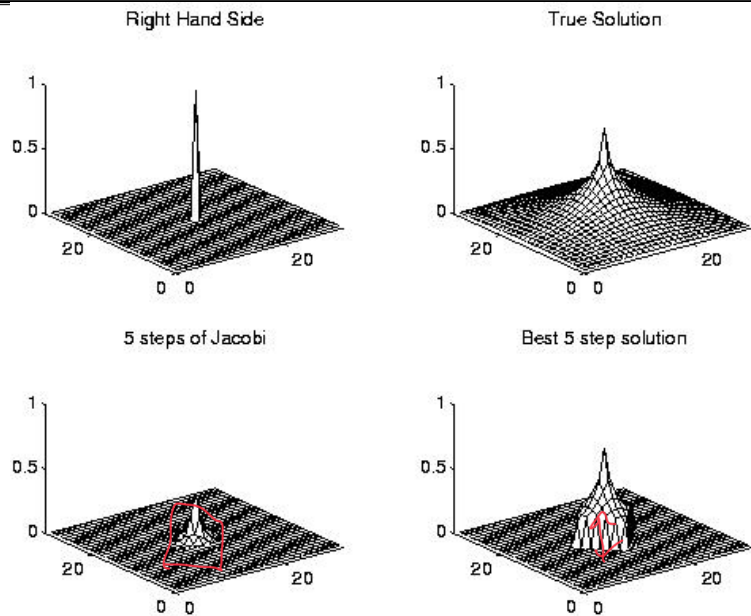
PRAM is an idealized parallel model with zero cost communication

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Multigrid Motivation



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Multigrid Motivation

- Recall that Jacobi, SOR, CG, or any other sparse-matrix-vector-multiply-based algorithm can only move information one grid cell at a time
 - Take at least n steps to move information across $n \times n$ grid
- Can show that decreasing error by fixed factor $c < 1$ takes $\Omega(\log n)$ steps
 - Convergence to fixed error < 1 takes $\Omega(\log n)$ steps
- Therefore, converging in $O(1)$ steps requires moving information across grid faster than to one neighboring grid cell per step
 - One step can't just do sparse-matrix-vector-multiply

Big Idea used in multigrid and elsewhere

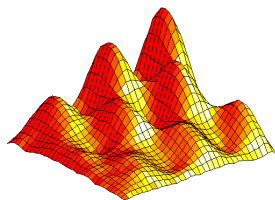
- **If you are far away, problem looks simpler**
 - For gravity: approximate earth, distant galaxies, ... by point masses
- **Can solve such a coarse approximation to get an approximate solution, iterating if necessary**
 - Solve coarse approximation problem by using an even coarser approximation of it, and so on recursively
- **Ex: Multigrid for solving PDE in $O(n)$ time**
 - Use coarser mesh to get approximate solution of Poisson's Eq.
- **Ex: Fast Multipole Method, Barnes-Hut for computing gravitational forces on n particles in $O(n \log n)$ time:**
 - Approximate particles in box by total mass, center of gravity
 - Good enough for distant particles; for close ones, divide box recursively
- **Ex: Graph Partitioning (used to parallelize SpMV)**
 - Replace graph to be partitioned by a coarser graph (CS267 for details)

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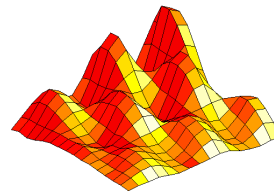
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Fine and Coarse Approximations



Fine



Coarse

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Multigrid Overview

- **Basic Algorithm:**
 - Replace problem on fine grid by an approximation on a coarser grid
 - Solve the coarse grid problem approximately, and use the solution as a starting guess for the fine-grid problem, which is then iteratively updated
 - Solve the coarse grid problem **recursively**, i.e. by using a still coarser grid approximation, etc.
- **Success depends on coarse grid solution being a good approximation to the fine grid**



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Multigrid uses Divide-and-Conquer in 2 Ways

- **First way:**
 - Solve problem on a given grid by calling Multigrid on a coarse approximation to get a good guess to refine
- **Second way:**
 - Think of error as a sum of sine curves of different frequencies
 - Same idea as FFT solution, but not explicit in algorithm
 - Each call to Multigrid responsible for suppressing coefficients of sine curves of the lower half of the frequencies in the error (pictures later)

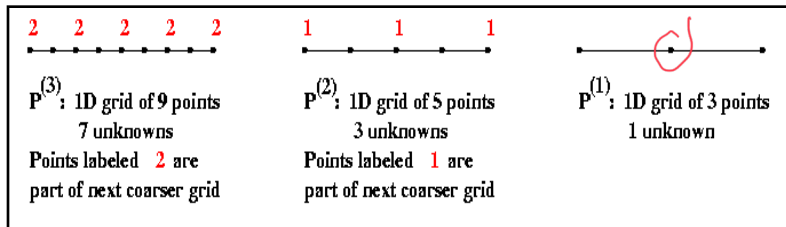
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Multigrid Sketch in 1D

- Consider a 2^{m+1} grid in 1D for simplicity
- Let $P^{(i)}$ be the problem of solving the discrete Poisson equation on a 2^{i+1} grid in 1D. Write linear system as $T(i) * x(i) = b(i)$
- $P^{(m)}, P^{(m-1)}, \dots, P^{(1)}$ is sequence of problems from finest to coarsest



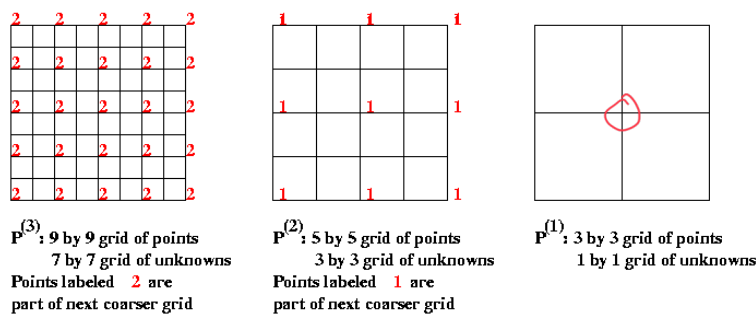
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Multigrid Sketch (1D and 2D)

- Consider a 2^{m+1} grid in 1D (2^{m+1} by 2^{m+1} grid in 2D) for simplicity
- Let $P^{(i)}$ be the problem of solving the discrete Poisson equation on a 2^{i+1} grid in 1D (2^{i+1} by 2^{i+1} grid in 2D)
 - Write linear system as $T(i) * x(i) = b(i)$
- $P^{(m)}, P^{(m-1)}, \dots, P^{(1)}$ is sequence of problems from finest to coarsest



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Multigrid Operators (write on board)

- For problem $P^{(i)}$:
 - $b(i)$ is the RHS and
 - $x(i)$ is the current estimated solution
- } both live on grids of size 2^{i-1}
- All the following operators just average values on neighboring grid points (so information moves fast on coarse grids)
- The **restriction operator $R(i)$** maps $P^{(i)}$ to $P^{(i-1)}$
 - Restricts problem on fine grid $P^{(i)}$ to coarse grid $P^{(i-1)}$
 - Uses sampling or averaging
 - $b(i-1) = R(i)(b(i))$
- The **interpolation operator $ln(i-1)$** maps approx. solution $x(i-1)$ to $x(i)$
 - Interpolates solution on coarse grid $P^{(i-1)}$ to fine grid $P^{(i)}$
 - $x(i) = ln(i-1)(x(i-1))$
- The **solution operator $S(i)$** takes $P^{(i)}$ and improves solution $x(i)$
 - Uses “weighted” Jacobi or SOR on single level of grid
 - $x \text{ improved } (i) = S(i)(b(i), x(i))$
- Overall algorithm, then details of operators

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Multigrid V-Cycle Algorithm (write on board, Matlab code on webpage)

Function MGv ($b(i)$, $x(i)$)

... Solve $T(i)*x(i) = b(i)$ given $b(i)$ and an initial guess for $x(i)$

... return an improved $x(i)$

if ($i = 1$)

compute exact solution $x(1)$ of $P^{(1)}$

only 1 unknown

return $x(1)$

else

$x(i) = S(i)(b(i), x(i))$

improve solution by
damping high frequency error
compute residual

$r(i) = T(i)*x(i) - b(i)$

$d(i) = ln(i-1)(MGV(R(i)(r(i)), 0))$

solve $T(i)*d(i) = r(i)$ recursively
correct fine grid solution

$x(i) = x(i) - d(i)$

$x(i) = S(i)(b(i), x(i))$

improve solution again

return $x(i)$

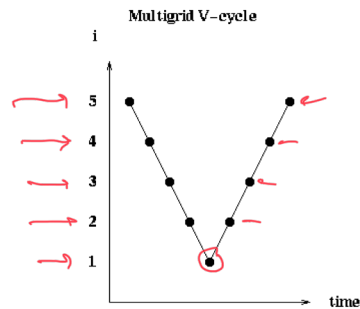
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Why is this called a V-Cycle?

- Just a picture of the call graph
- In time a V-cycle looks like the following



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Cost (#flops) of a V-Cycle for 2D Poisson

- Constant work per mesh point (average with neighbors)
- Work at each level in a V-cycle is $O(\text{the number of unknowns})$
- Cost of Level i is $O((2^i - 1)^2) = O(4^i)$
- If finest grid level is m , total time is:

$$\sum_{i=1}^m O(4^i) = O(4^m) = O(\text{\# unknowns})$$

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Full Multigrid (FMG)

◦ Intuition:

- improve solution by doing multiple V-cycles
- avoid expensive fine-grid (high frequency) cycles
- analysis of why this works is beyond the scope of this class

Function FMG (b(m), x(m))

... return improved x(m) given initial guess

compute the exact solution x(1) of P(1) ←

for i=2 to m

$$x(i) = \text{MGV} (b(i), \text{In} (i-1) (x(i-1)))$$

◦ In other words:

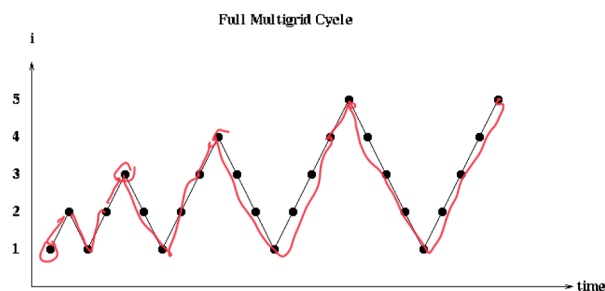
- Solve the problem with 1 unknown
- Given a solution to the coarser problem, $P^{(i-1)}$, map it to starting guess for $P^{(i)}$
- Solve the finer problem using the Multigrid V-cycle

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Full Multigrid Cost Analysis



◦ One V-cycle for each call to FMG

- people also use “W cycles” and other compositions

◦ #Flops: $\sum_{i=1}^m O(4^i) = O(4^m) = O(\# \text{ unknowns})$

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Complexity of Solving Poisson's Equation

- Theorem: error after one FMG call $\leq c \cdot$ error before, where $c < 1/2$, independent of # unknowns
- Corollary: We can make the error $<$ any fixed tolerance in a fixed number of steps, independent of size of finest grid
- This is the most important convergence property of MG, distinguishing it from all other methods, which converge more slowly for large grids
- Total complexity just proportional to cost of one FMG call

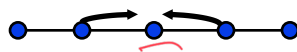
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The Solution Operator $S(i)$ – Details (on board)

- The solution operator, $S(i)$, is a weighted Jacobi
- Consider the 1D problem



- At level i , pure Jacobi replaces:

$$x(j) := \frac{1}{2} (x(j-1) + x(j+1) + b(j))$$

$$\text{in notation from lecture: } \underline{R_J} = I - \underline{T/2}$$

- Weighted Jacobi uses:

$$x(j) := \frac{1}{3} (x(j-1) + x(j) + x(j+1) + b(j))$$

$$\underline{R_{wJ}} = I - \underline{T/3}$$

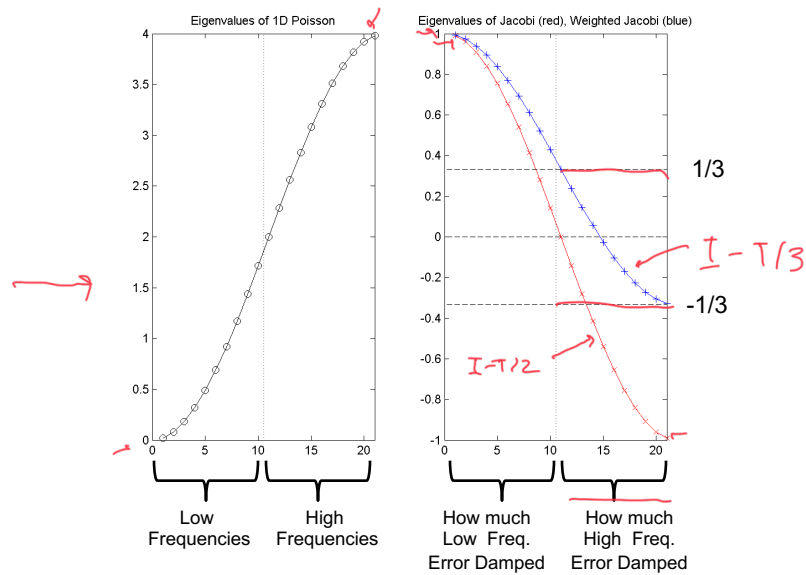
- In 2D, similar average of nearest neighbors
 - Chosen so that “high frequency” eigenvector components of error get decreased by as much as possible ($1/3$)

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Eigenvalues of Solution Operator $S(i)$ ($R_{wJ} = I - T/3$)

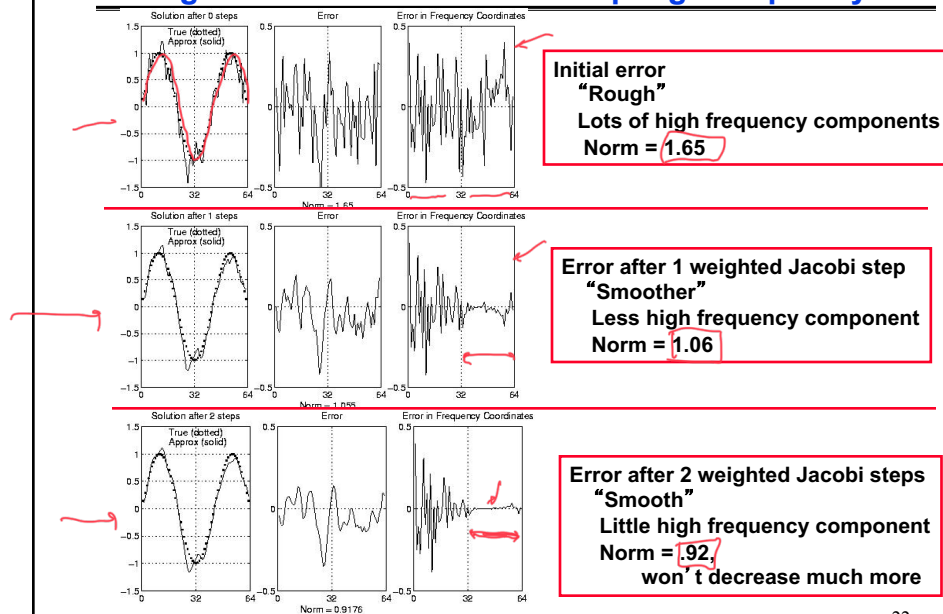


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Weighted Jacobi chosen to damp high frequency error



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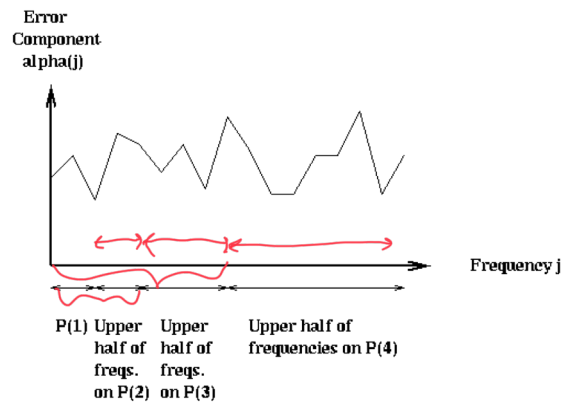
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Multigrid as Divide and Conquer Algorithm

- Each level in a V-Cycle reduces the error in one part of the frequency domain

Schematic Description of Multigrid

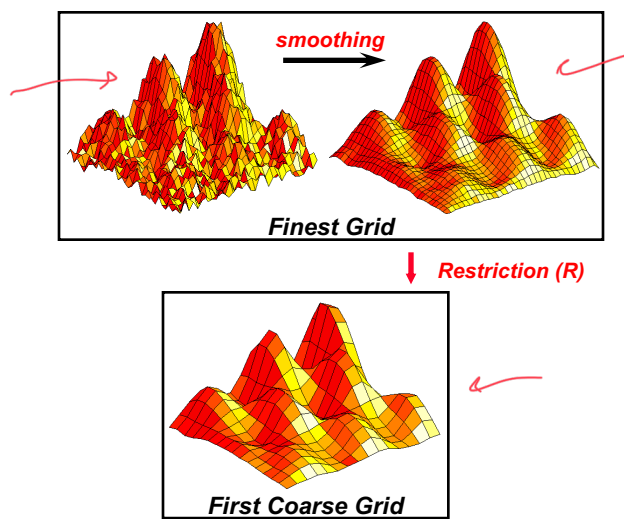


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Error on fine and coarse grids



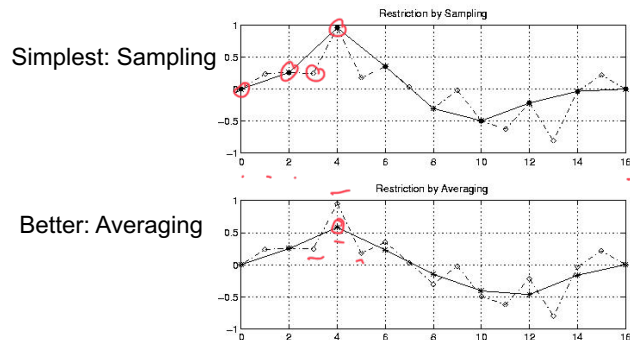
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The Restriction Operator $R(i)$ - Details

- The restriction operator, $R(i)$, takes
 - a problem $P^{(i)}$ with Right-Hand-Side (RHS) b_{fine} and
 - maps it to a coarser problem $P^{(i-1)}$ with RHS $b_{\text{coarse}} = R(i)(b_{\text{fine}})$
- In 1D, average values of neighbors
 - Simplest: Sampling: $b_{\text{coarse}}(k) = b_{\text{fine}}(k)$
 - Better: Averaging: $b_{\text{coarse}}(k) = 1/4 * b_{\text{fine}}(k-1) + 1/2 * b_{\text{fine}}(k) + 1/4 * b_{\text{fine}}(k+1)$



- In 2D, average with all 8 neighbors (N,S,E,W,NE,NW,SE,SW)

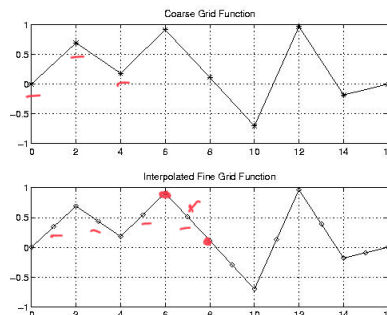
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Interpolation Operator $ln(i-1)$: details

- The interpolation operator $ln(i-1)$, takes a function x_{coarse} on a coarse grid $P^{(i-1)}$, and produces a function x_{fine} on a fine grid $P^{(i)}$:
 - $x_{\text{fine}} = ln(i-1)(x_{\text{coarse}})$
- In 1D, linearly interpolate nearest coarse neighbors
 - $x_{\text{fine}}(k) = x_{\text{coarse}}(k)$ if the fine grid point k is also a coarse one, else
 - $x_{\text{fine}}(k) = 1/2 * x_{\text{coarse}}(\text{left of } k) + 1/2 * x_{\text{coarse}}(\text{right of } k)$



coarse

fine

- In 2D, interpolation requires averaging with 4 nearest neighbors (NW,SW,NE,SE)

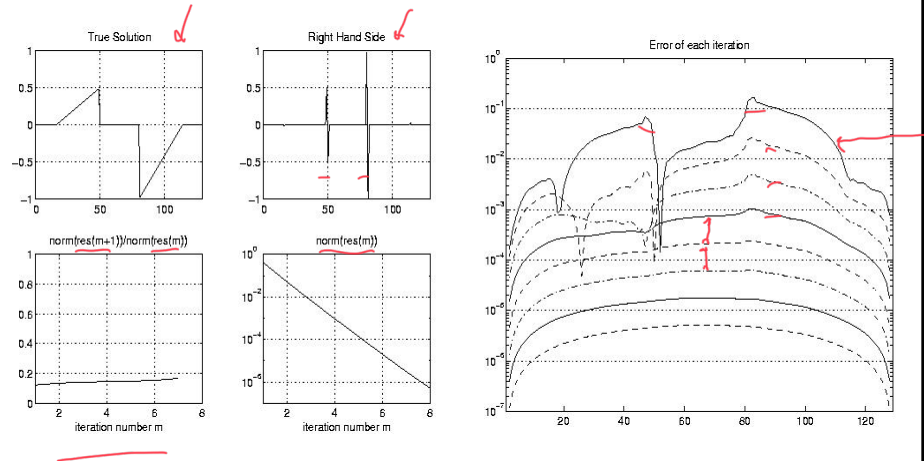
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Convergence Picture of Multigrid in 1D

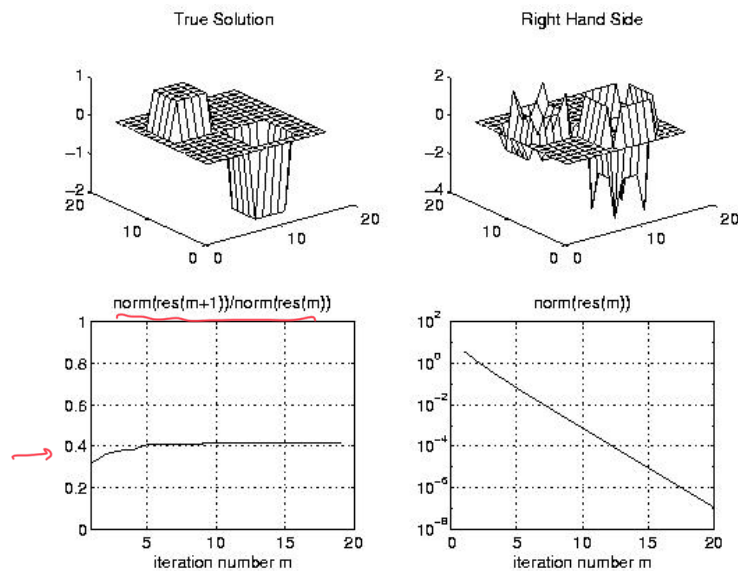


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Convergence Picture of Multigrid in 2D



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Multigrid V-Cycle Algorithm Analysis (1/2)

Function MG ($b(i)$, $x(i)$)

... Solve $T(i) \cdot x(i) = b(i)$ given $b(i)$ and an initial guess for $x(i)$

... return an improved $x(i)$

if ($i = 1$)

compute exact solution $x(1)$ of $P^{(1)}$ **only 1 unknown**

return $x(1)$

else

$x(i) = S(i) (b(i), x(i))$

$x(i) = S \cdot x(i) + b(i)/3$

$r(i) = T(i) \cdot x(i) - b(i)$

$r(i) = T(i) \cdot x(i) - b(i)$

$d(i) = \ln(i-1) (\text{MGV} (R(i) (r(i)), 0))$

$d(i) = \ln \cdot (T(i-1)^{-1} \cdot (R \cdot r(i)))$

(Note: we assume recursive solve is exact, for ease of analysis)

$x(i) = x(i) - d(i)$

$x(i) = x(i) - d(i)$

$x(i) = S(i) (b(i), x(i))$

$x(i) = S \cdot x(i) + b(i)/3$

return $x(i)$

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Multigrid V-Cycle Algorithm Analysis (2/2)

Goal: combine these equations to get formula for error $e(i) = x(i) - x$:

$x(i) = S \cdot x(i) + b(i)/3$

subtract $x = S \cdot x + b(i)/3$ to get $e(i) = S \cdot e(i)$

$r(i) = T(i) \cdot x(i) - b(i)$

subtract $0 = T(i) \cdot x - b(i)$ to get $r(i) = T(i) \cdot e(i)$

$d(i) = \ln \cdot (T(i-1)^{-1} \cdot (R \cdot r(i)))$

assume coarse problem solved exactly

$x(i) = x(i) - d(i)$

subtract $x = x$ to get $e(i) = e(i) - d(i)$

$x(i) = S \cdot x(i) + b(i)/3$

subtract $x = S \cdot x + b(i)/3$ to get $e(i) = S \cdot e(i)$

Substitute each equation into later ones to get

$e(i) = S \cdot (I - \ln \cdot (T(i-1)^{-1} \cdot (R \cdot T(i)))) \cdot S \cdot e(i) \equiv M \cdot e(i)$

Theorem: For 1D Poisson problem, the eigenvalues of M are either 0 or $1/9$, independent of dimension.

This means multigrid converges in a bounded number of steps, independent of dimension.

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Generalizing Multigrid beyond Poisson, to unstructured meshes (1/2)

- What does it mean to do Multigrid anyway?
- Need to be able to coarsen grid (hard problem)
 - Can't just pick "every other grid point" anymore
 - How to make coarse graph approximate fine one
 - What if there are no grid points?
- Need to define $R()$ and $In()$
 - How do we convert from coarse to fine mesh and back?
 - How do we define coarse matrix (no longer formula, like Poisson)
- Need to define $S()$
 - How do we damp "high frequency" error?
- Dealing with coarse meshes efficiently
 - Should we switch to another solver on coarsest meshes?

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Generalizing Multigrid beyond Poisson, to unstructured meshes (2/2)

- Given original problem, how do we generate a sequence of coarse approximations?
- For finite element problems, could regenerate matrix starting on coarser mesh
 - Need access to original physical problem and finite element modeling system, i.e. a lot more than just the original matrix, so it may be impossible
 - What does "coarse" mean, once very coarse?
- Geometric Multigrid
 - Assume we know (x,y,z) coordinates of underlying mesh, and matrix
 - Generate coarse mesh points, analogous to taking every other point in regular mesh
 - Retriangulate to get new mesh
 - Use finite element shape functions on coarse mesh to project fine matrix to coarse one
- Algebraic Multigrid
 - Don't even have (x,y,z) coordinates, just matrix

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Geometric Multigrid

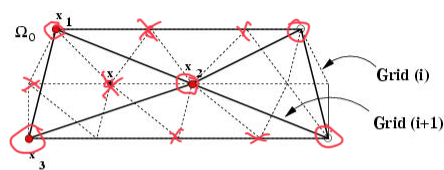
- **Need matrix, (x,y,z) coordinates of mesh points**
 - Not minimum information (just matrix), but a little more
 - Based on work of Guillard, Chan, Smith
- **Finite element intuition**
 - Goal is to compute function, represented by values at points
 - Think of approximation by piecewise linear function connecting points
 - Easy in 1D, need triangulated mesh in 2D, 3D uses tetrahedra
- **Geometric coarsening**
 - • Pick a subset of coarse points “evenly spaced” among fine points
 - Use Maximal Independent Sets
 - Try to keep important points, like corners, edges of object
 - • Retriangulate coarse points
 - Try to approximate answer by piecewise linear function on new triangles
 - • Let columns of P (“prolongator”) be values at fine grid points given values at coarse ones
 - Generalizes Interpolation operator “ In ” from before
 - $A_{\text{coarse}} = P^T A_{\text{fine}} P$ – Galerkin method
 - For Poisson: $P = In$, $P^T = 2^*R$, $T_{\text{coarse}} = 2^* P^T * T_{\text{fine}} * P$

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Example of Geometric Coarsening



Simple Greedy Algorithm:

```

repeat
  pick unmarked vertex
  mark it and its neighbors
until no unmarked vertices
  
```

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Examples of meshes from geometric coarsening

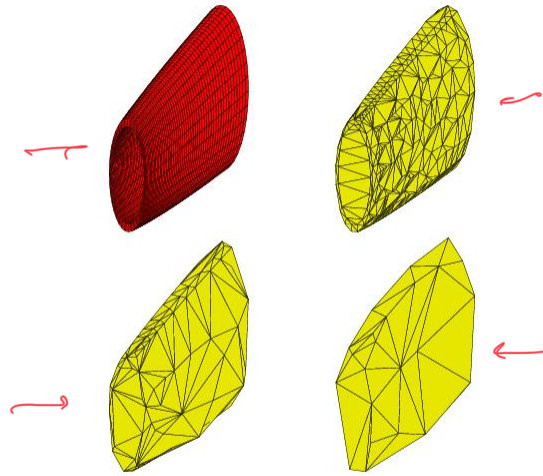


Figure 6: Sample input grid and coarse grids

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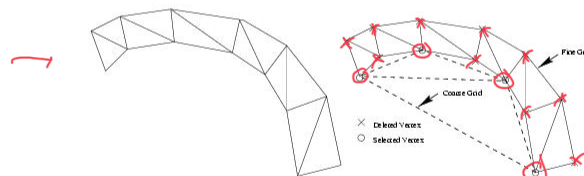
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What can go wrong

- Care needed so coarse grid preserves geometric features of fine grid
 - Label fine grid points as corner, edge, face, interior
 - Delete edges between same-labeled points in different features
 - Ex: delete edges between points on different faces
 - Keeps feature represented on coarse meshes

Pathological example:



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How to coarsen carefully

Example - classify vertices - modify graph

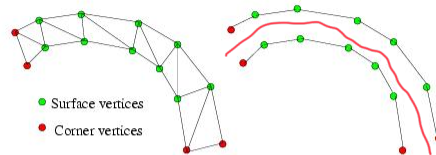


Figure 1: Modify graph

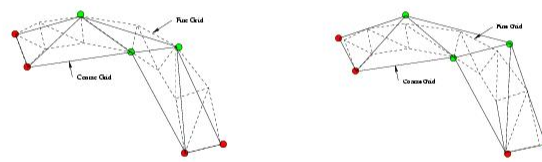


Figure 2: New mesh - fixed mesh

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Algebraic Multigrid

- No information beyond matrix needed
- Galerkin still used to get $A_{\text{coarse}} = P^T A_{\text{fine}} P$
- Prolongator P defined purely algebraically
 - Cluster fine grid points into nearby groups
 - Can use Maximal Independent Sets or Graph Partitioning
 - Use magnitude of entries of A_{fine} to cluster
 - Associate one coarse grid node to each group
 - To interpolate coarse grid values to associated fine grid point, can use properties of PDE, eg elasticity:
 - Rigid body modes of coarse grid point
 - Let coarse grid point have 6 dof (3 translation, 3 rotation)
 - Can be gotten from QR factorization of submatrix of A_{fine}
 - Can also apply smoother to resulting columns of P
 - "Smoothed Aggregation"
- Based on work of Vanek, Mandel, Brezina, Farhat, Roux, Bulgakov, Kuhn ...

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Parallel Smoothers for Unstructured Multigrid

- **Weighted Jacobi**
 - Easy to implement, hard to choose weight
- **Gauss-Seidel**
 - Works well, harder to parallelize because of triangular solve
- **Polynomial Smoothers**
 - Chebyshev polynomial $p(A_{\text{fine}})$
 - Easy to implement (just SpMV with A_{fine})
 - Chebyshev chooses $p(y)$ such that
 - $|1 - p(y)| = \min$ over interval $[\lambda^*, \lambda_{\max}]$ estimated to contain eigenvalues of A_{fine}

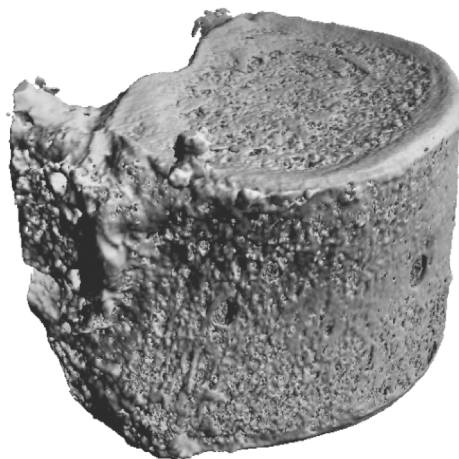
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Source of Unstructured Finite Element Mesh: Vertebra

Study failure modes of trabecular Bone under stress



Source: M. Adams, H. Bayraktar, T. Keaveny, P. Papadopoulos, A. Gupta

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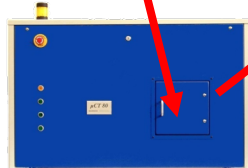
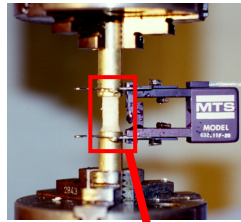
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Methods: μ FE modeling

Mechanical Testing

E , ϵ_{yield} , σ_{ult} , etc.

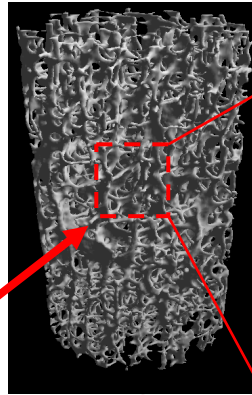
Source: Mark Adams, PPPL



Micro-Computed Tomography
 μ CT @ 22 μm resolution

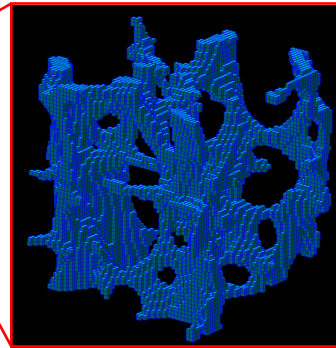
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3D image



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μ FE mesh
2.5 mm cube
44 μm elements

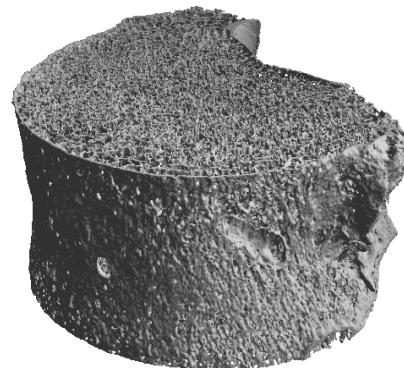


Up to 537M unknowns

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Vertebral Body With Shell

- Large deformation elasticity
- 6 load steps (3% strain)
- Scaled speedup
 - ~131K dof/processor
- 7 to 537 million dof
- 4 to 292 nodes
- IBM SP Power3
 - 14 of 16 procs/node used
 - Up to 4088 processors
- Double/Single Colony switch
- Gordon Bell Prize, 2004
- Clinical application to predicting chance of fracture due to osteoporosis

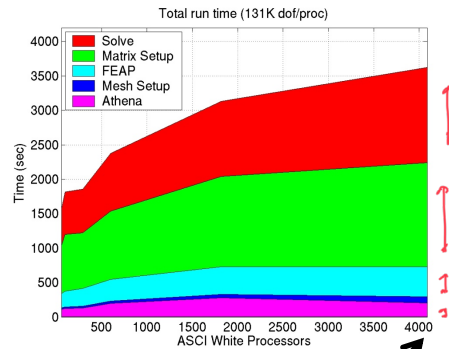
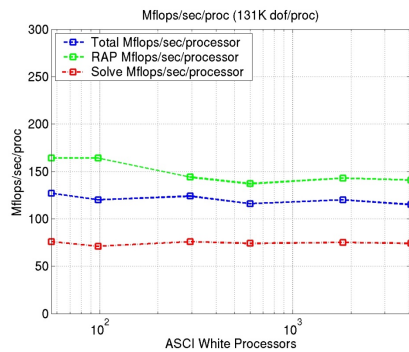


80 μm w/ shell

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131K dof / proc (weak scaling) - Flops/sec/proc .47 Teraflops - 4088 processors



537M dof !

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Conclusions

- **Multigrid can be very fast**
 - Provably “optimal” (does $O(N)$ flops to compute N unknowns) for many problems in which one can show that using a coarse grid gives a good approximation
 - Can be parallelized effectively
- **Multigrid can be complicated to implement**
 - Lots of software available (see web page for pointers)
 - PETSc (includes many iterative solvers, interfaces to other packages, Python interface, runs in parallel)
 - ACTS (repository for PETSc and other packages)
 - Offers periodic short courses on using these packages
 - MGNET
 - Sample Matlab implementation for 1D and 2D Poisson
 - See class web page under “Matlab Programs for Homework Assignments”

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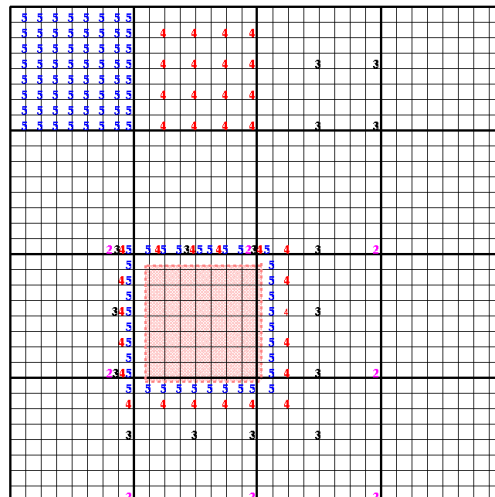
Extra slides

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Parallel 2D Multigrid

- Multigrid on 2D requires nearest neighbor (up to 8) computation at each level of the grid
- Start with $n=2^m+1$ by 2^m+1 grid (here $m=5$)
- Use an s by s processor grid (here $s=4$)



Communication pattern for Multigrid on 33 by 33 mesh with 4 by 4 processor grid
 In top processor row, grid points labeled m are updated in problem $P(m)$ of multigrid
 Pink processor owns grid points inside pink box
 In lower half of graph, grid points labeled m need to be communicated to pink processor
 in problem $P(m)$ of multigrid

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Performance Model of parallel 2D Multigrid (V-cycle)

- Assume 2^{m+1} by 2^{m+1} grid of points, $n = 2^m - 1$, $N = n^2$
- Assume $p = 4^k$ processors, arranged in 2^k by 2^k grid
 - Processors start with 2^{m-k} by 2^{m-k} subgrid of unknowns
- Consider V-cycle starting at level m
 - At levels m through k of V-cycle, each processor does some work
 - At levels $k-1$ through 1 , some processors are idle, because a 2^{k-1} by 2^{k-1} grid of unknowns cannot occupy each processor
- Cost of one level in V-cycle
 - If level $j \geq k$, then cost =
 - $O(4^{j-k})$ Flops, proportional to the number of grid points/processor
 - $+ O(1) \alpha$ Send a constant # messages to neighbors
 - $+ O(2^{j-k}) \beta$ Number of words sent
 - If level $j < k$, then cost =
 - $O(1)$ Flops, proportional to the number of grid points/processor
 - $+ O(1) \alpha$ Send a constant # messages to neighbors
 - $+ O(1) \beta$ Number of words sent
- Sum over all levels in all V-cycles in FMG to get complexity

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Comparison of Methods (in $O(\cdot)$ sense)

	# Flops	# Messages	# Words sent
MG	$N/p + \log p * \log N$	$(\log N)^2$	$(N/p)^{1/2} + \log p * \log N$
FFT	$N \log N / p$	$p^{1/2}$	N/p
SOR	$N^{3/2} / p$	$N^{1/2}$	N/p

- SOR is slower than others on all counts
- Flops for MG and FFT depends on accuracy of MG
- MG communicates less total data (bandwidth)
- Total messages (latency) depends ...
 - This coarse analysis can't say whether MG or FFT is better when $\alpha \gg \beta$

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Practicalities

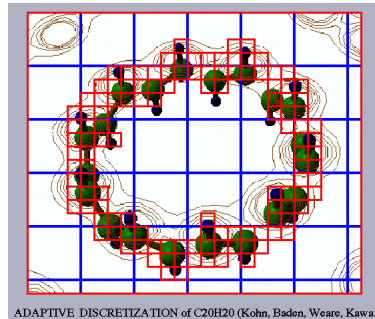
- In practice, we don't go all the way to $P^{(1)}$
- In sequential code, the coarsest grids are negligibly cheap, but on a parallel machine they are not.
 - Consider 1000 points per processor
 - In 2D, the surface to communicate is $4\sqrt{1000} \approx 128$, or 13%
 - In 3D, the surface is $1000^{2/3} \approx 500$, or 50%
- See Tuminaro and Womble, SIAM J. Sci. Comp., v14, n5, 1993 for analysis of MG on 1024 nCUBE2
 - on 64x64 grid of unknowns, only 4 per processor
 - efficiency of 1 V-cycle was .02, and on FMG .008
 - on 1024x1024 grid
 - efficiencies were .7 (MG Vcycle) and .42 (FMG)
 - although worse parallel efficiency, FMG is 2.6 times faster than V-cycles alone
 - nCUBE had fast communication, slow processors

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Multigrid on an Adaptive Mesh

- For problems with very large dynamic range, another level of refinement is needed
- Build adaptive mesh and solve multigrid (typically) at each level
- Can't afford to use finest mesh everywhere

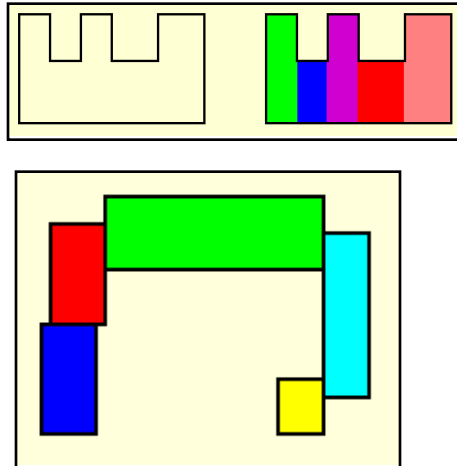


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Multiblock Applications

- Solve system of equations on a union of rectangles
 - subproblem of AMR
- E.g.,

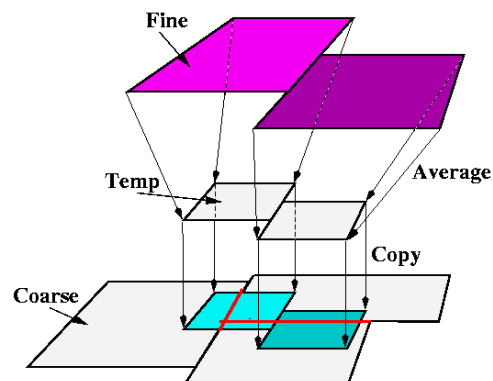


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Adaptive Mesh Refinement

- Data structures in AMR
- Usual parallelism is to assign grids on each level to processors
- Load balancing is a problem

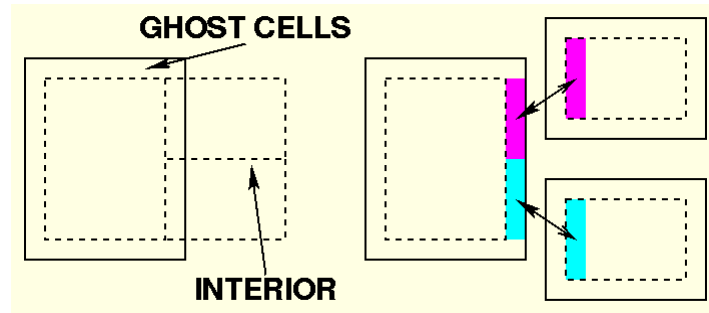


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Support for AMR

- Domains in Titanium designed for this problem
- Kelp, Boxlib, and AMR++ are libraries for this
- Primitives to help with boundary value updates, etc.

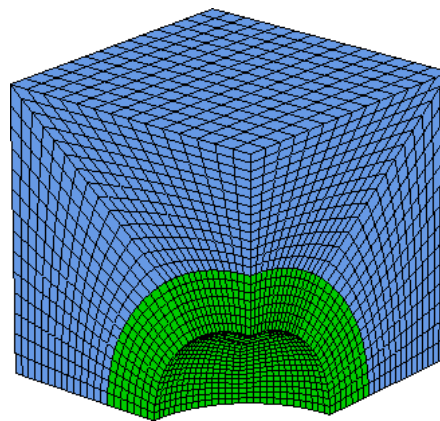


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Multigrid on an Unstructured Mesh

- Another approach to variable activity is to use an unstructured mesh that is more refined in areas of interest
- Adaptive rectangular or unstructured?
 - Numerics easier on rectangular
 - Supposedly easier to implement (arrays without indirection) but boundary cases tend to dominate code



Up to 39M unknowns on 960 processors,
With 50% efficiency (Source: M. Adams)

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