Chap 6+7: Iterative Methods
Part 4: Splitting Methods
Convergence Analysis

Def: Splitting of $A = M - K$, M nonsingular

$Ax = b \Rightarrow Mx = Kx + b$

$\Rightarrow$ iterate $Mx_{i+1} = Kx_i + b$

$\Rightarrow x_{i+1} = M^{-1}Kx_i + M^{-1}b = Rx_i + c$

Def spectral radius $\rho(R) = \max_i |\lambda_i(R)|$

Thm: $x_{i+1} = Rx_i + c$ converges to solution of $Ax = b$ for all x_0 if and only if $\rho(R) < 1$

Define Jacobi, Gauss-Seidel (GS)
Successive Overrelaxation (SOR)

$A = D - L' - U'$ = diagonal + strict lower triangle + strict upper triangle

$= D(I - L - U)$

Jacobi: $A = M - K = D - (L' + U')$

$\Rightarrow R_J = M^{-1}K = L + U$

GS: $A = M - K = (D - L') - U'$

$\Rightarrow R_{GS} = (D - L')^{-1}U' = (I - L)^{-1}U$

SOR: $x_{i+1}^{SOR}(\omega) = (1-\omega)x_i + \omega \cdot x_{i+1}^{GS}$

$A = M - K = (\frac{1}{\omega}D - L') - (-D + \frac{1}{\omega}D + U')$

$R_{SOR}(\omega) = (I - \omega L)^{-1}((1-\omega)I + \omega U)$

Model Problem:

T_n : 1D model problem

$\lambda_i = 2(1 - \cos\frac{\pi i}{n+1})$

$T_{n\times n}$: 2D model problem

eigenvalues $\lambda_i + \lambda_j$ forall (i, j)

Jacobi for Model Problem:

$T_{n\times n} = M - K = 4 \cdot I - (4I - T_{n\times n})$

$R_J = M^{-1}K = I - \frac{1}{4}T_{n\times n}$

Eigenvalues of R_J are $1 - \frac{1}{4}(\lambda_i + \lambda_j)$

so $\rho(R_J) = 1 - \frac{\lambda_{min}}{2} = 1 - (1 - \cos\frac{\pi}{n+1})$

$= \cos\frac{\pi}{n+1} \sim 1 - \frac{\pi^2}{2(n+1)^2}$

large $n \Rightarrow \rho(R_J)$ close to 1

$\Rightarrow$ slower convergence

Error after m steps of Jacobi multiplied by $\rho(R_J)^m$

$\rho(R_J) = 1 - x$, solve $(1-x)^m = e^{-1}$ for simplicity:

$$(1-x)^m = (1-x)^{\frac{1}{x}\cdot mx} \sim e^{-mx} = e^{-1}$$

$\Rightarrow m = \frac{1}{x}$ steps needed

2D model problem $\frac{1}{x} = \frac{2(n+1)^2}{\pi^2} = O(n^2) = O(N)$

$N = n^2 = \#$ unknowns

$\Rightarrow \#$ iteration proportional to N

Total cost of Jacobi for 2D model problem

$= O(N)$ iterations $\cdot\, O(N)$ flops per iteration

$= O(N^2)$

Typical for many iterative methods

i.e. the number of iteration grows with N

: One important exception: Multigrid

GS: provided we update solution components in right order, for model problem $\rho(R_{GS}) = \rho(R_J)^2$

$\Rightarrow$ 1 step of GS as good as 2 steps of Jacobi

$\Rightarrow$ same $O(\ )$ cost

SOR(ω): with right update order can choose optimal ω_{OPT}

$\rho(R_{SOR(\omega_{OPT})}) \sim 1 - \frac{2\pi}{n}$

for large n, "much farther" from 1 than $\rho(R_J)$ or $\rho(R_{GS})$

only need $O(n) = O(\sqrt{N})$ to reduce error by constant factor

$\Rightarrow$ overall cost $= O(N^{3/2})$ vs. $O(N^2)$ for Jacobi or SOR

Next: present and prove some of general theory

Thm 1: If A is strictly row diagonally dominant:

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}|$$

Then both Jacobi and GS converge and GS at least as fast as Jacobi

$$\|R_{GS}\|_\infty \leq \|R_J\|_\infty < 1$$

proof (Just for Jacobi, see Thm 6.2 for full proof)

$$A = D - (D-A) \Rightarrow R_J = D^{-1}(D-A) = I - D^{-1}A$$

$$\sum_i |R_J(j,i)| = \underbrace{\left|1 - \frac{A(j,j)}{A(j,j)}\right|}_{=0} + \sum_{i \neq j} \frac{|A(j,i)|}{|A(j,j)|}$$

$$= \sum_{i \neq j} \frac{|A(j,i)|}{|A(j,j)|}$$

< 1 by strict diagonal dominance

$$\Rightarrow \|R_J\|_\infty < 1 \Rightarrow \text{Jacobi converges}$$

Model Problem not strictly row diagonally dominant

(diagonal $=4$, most rows have
4 -1s in off diagonal)

Def: if A satisfies

$$|A(i,i)| \geq \sum_{j \neq i} |A(i,j)| \text{ in all rows}$$

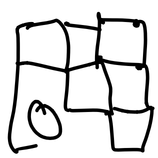
with strict inequality in at least 1 row, weakly row diagonally dominant

Not enough for convergence:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and R^i does not converge

Def: A is irreducible if there is no permutation matrix P such that PAP^T is block triangular

i.e. 

: In graph theory language

graph described by A is "strongly connected"
i.e. path from every vertex to every vertex

Model problem: Graph is a 2D mesh, and a mesh is strongly connected
also, for Model problem, for vertices next to boundary, only 2 or 3 -1s in that row $\Rightarrow T_{n \times n}$ is weakly row diagonally dominant

Thm 2: If A weakly row diagonally dominant and irreducible (like the model problem) then

$\rho(R_{GS}) < \rho(R_J) < 1$ so both converge, GS faster than Jacobi

(Thm 6.3 in text, proof in [249])

Thm 3: If A is symmetric pos def then SOR(w) converge if and only if $0 < w < 2$, in particular

SOR(1)=GS converges

(Thm 6.5 in text, proof in text)

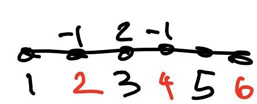
To analyze SOR(ω) on model problem need another graph theoretic property of matrix:

Def: Matrix has Property A: if there is a permutation P such that

$$PAP^T = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$ where A_{11} and A_{22} are diagonal

In graph theory terms: the vertices V of graph can be partitioned $V = V_1 \cup V_2$ where there are no edges connecting V_1 to V_1, or V_2 to V_2.

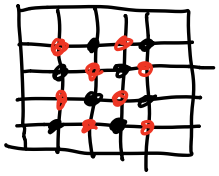
Such a graph is called bipartite

Ex: 1D Model problem 

V_1 = odd numbered nodes

V_2 = even numbered nodes

Ex: 2D Model problem



$V_1 = (i,j)$ where $i+j$ odd

$V_2 = (i,j)$ where $i+j$ even

Ex: Same idea for 3D Poisson etc

Thm 4: (Thm 6.6 in text)

Suppose A has property A and we do $SOR(w)$ updating all vertices in V_1 (black) first, and then all vertices in V_2 (red)

Then the eigenvalues μ of R_J and eigenvalues λ of $R_{SOR(w)}$ are related by

$$(*) \quad (\lambda + w - 1)^2 = \lambda w^2 \mu^2$$

In particular, when $w=1$, $\lambda = \mu^2$

so $\rho(R_{SOR(1)}) = \rho(R_{GS}) = \rho(R_J)^2$

ie. GS twice as fast as Jacobi

Proof: $A = \begin{array}{c} \\ v_1 \\ v_2 \end{array}\begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}$ A_{11} and A_{22} diagonal

2D Poisson $T_{n\times n} = 4I + \left[\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & 0 \end{array}\right] = D - L' - U'$

λ is an eigenvalue of $R_{SOR(w)}$

$$\Rightarrow 0 = \det(\lambda I - R_{SOR(w)})$$
$$= \det(\lambda I - (I - wL)^{-1}((1-w)I + wU))$$
$$= \det(\underbrace{(I - wL)}\,(\ \ ''\ \ ))$$
$$= \det(\lambda I - \lambda w L - (1-w)I - wU)$$
$$= \det((\lambda - 1 + w)I - \lambda w L - wU)$$
$$= \det(\underbrace{\sqrt{\lambda}\, w}((\frac{\lambda - 1 + w}{\sqrt{\lambda}\, w})I - \sqrt{\lambda} L - \frac{1}{\sqrt{\lambda}} U))$$
$$= (\sqrt{\lambda} w)^n \cdot \det((\frac{\lambda - 1 + w}{\sqrt{\lambda}\, w})I - \sqrt{\lambda} L - \frac{1}{\sqrt{\lambda}} U)$$

$$D = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \frac{1}{\sqrt{\lambda}} I \end{array}\right]$$

$$D(\sqrt{\lambda} L)D^{-1} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \frac{1}{\sqrt{\lambda}} I \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline \sqrt{\lambda} L_{21} & 0 \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \sqrt{\lambda} I \end{array}\right]$$
$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline L_{21} & 0 \end{array}\right] = L$$

$$D(\frac{1}{\sqrt{\lambda}} U)D^{-1} = U$$

determinant above

$$0 = (\sqrt{\lambda}\omega)^n \det\left(D(\ ''\ )D^{-1}\right)$$

$$= (\cancel{\sqrt{\lambda}\omega})^n \det\left(\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\omega}\right)I - L - U\right)$$

$$0 = \det\left(\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\omega}\right)I - R_J\right)$$

i.e. $\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} = \mu$, an eigenvalue of R_J

QED

More generally, since we know all eigenvalues μ of R_J for model problem, we can use (*) to write down all eigenvalues of $R_{SOR(\omega)}$, and then pick ω_{OPT} to minimize $\rho(R_{SOR(\omega_{OPT})})$

Thm 5 (Thm 6.7 in text)

Suppose A has Property A, do SOR(ω) updating in V_1 before V_2, all eigenvalues of R_J are real and $\mu = \rho(R_J) < 1$ (all true for model problem)

Then $\omega_{OPT} = \dfrac{2}{1+\sqrt{1-\nu^2}}$

$$\rho(SOR(\omega_{OPT})) = \omega_{OPT} - 1$$

$$= \frac{\nu^2}{\left(1+\sqrt{1-\nu^2}\right)^2}$$

In particular, for 2D model problem, on $n \times n$ grid:

$$\omega_{OPT} = \frac{2}{1+\sin\frac{\pi}{n+1}} \sim 2$$

$$\rho(SOR(\omega_{OPT})) = \frac{\cos^2\left(\frac{\pi}{n+1}\right)}{\left(1+\sin\left(\frac{\pi}{n+1}\right)\right)^2}$$

$$\sim 1 - \frac{2\pi}{n+1}$$

proof follows by solving (*) for λ