Welcome to Ma221! Lecture 23, Fall24

Splitting Methods for $Ax=b$

Def: $A = M - K$, M nonsingular

$$Ax=b \rightarrow Mx = Kx + b$$

$$\rightarrow Mx_{i+1} = Kx_i + b$$

$$(*) \qquad x_{i+1} = M^{-1}Kx_i + M^{-1}b$$

$$= Rx_i + c$$

Def: spectral radius of $R = \rho(R) = \max_{\lambda \text{ eval of } R} |\lambda|$

Thm: $(*)$ converges to $A^{-1}b$ for all x_0 if and only if $\rho(R) < 1$

$$A = \begin{bmatrix} D & & -U' \\ & D & \\ -L' & & D \end{bmatrix} = D - L' - U' = D(I - L - U)$$

Jacobi, in words:

for $j = 1$ to n pick $x_{i+1}(j)$ to exactly solve equation j

as a splitting: $A = M - K = D - (L' + U')$

$$R_J = D^{-1}(L' + U') = L + U$$

Gauss-Seidel, in words:

Improve on Jacobi by using using most recently updated values of x

as a splitting: $A = (D-L') - U' = M-K$

$$R_{GS} = M^{-1}K = (D-L')^{-1}U'$$
$$= (D(I-L))^{-1}U'$$
$$= (I-L)^{-1}U$$

SOR(w): Successive Overrelaxation:

In words: weighted linear combination of x_i and result of GS

$$x_{i+1}^{SOR(w)}(j) = (1-w)\cdot x_i(j) + w\cdot x_{i+1}^{GS}(j)$$

$w=1 \Rightarrow$ same as GS

$w<1 \Rightarrow$ "underrelaxation", not useful

$w>1 \Rightarrow$ "overrelaxation", go further in same direction as GS

Later: how to choose w optimally for Poisson

As a loop

for $j=1:n$

$$x_{i+1}(j) = (1-w)x_i(j) + w\Big(b_j - \sum_{k<j} A(j,k)x_{i+1}(j) - \sum_{k>j} A(j,k)x_i(j)\Big)/A(j,j)$$

As a splitting: Multiply inner loop by $A(j,j)$

$$(D - \omega L')x_{i+1} = ((I-\omega)Dx_i + \omega U' x_i) + \omega b$$

divide by ω

$$A = \left(\tfrac{1}{\omega}D - L'\right) - \left(\tfrac{1}{\omega}D - D + U'\right)$$
$$= \quad M \quad - \quad K$$

$$R^{SOR(\omega)} = M^{-1}K = \left(\tfrac{1}{\omega}D - L'\right)^{-1}\left(\tfrac{1}{\omega}D - D + U'\right)$$
$$= (I - \omega L)^{-1}((1-\omega)I + \omega U)$$

For 2D Poisson, red-black ordering

red $\equiv$ $j+k$ even, black $\equiv$ $j+k$ odd

for all reds:

$$V_{i+1}(j,k) = (1-\omega)V_i(j,k) + \omega\underbrace{(V_i(j-1,k) + V_i(j+1,k) + V_i(j,k-1) + V_i(j,k+1) + h^2F(j,k))/4}_{\text{old black data}}$$

for all blacks:

$$V_{i+1}(j,k) = (1-\omega)V_i(j,k) + \omega\underbrace{(V_{i+1}(j-1,k) + V_{i+1}(j+1,k) + V_{i+1}(j,k-1) + V_{i+1}(j,k+1) + h^2F(j,k))/4}_{\text{updated red data}}$$

Convergence of Splitting Methods In general, and for 2D Poisson

Jacobi: $T_{n\times n} = M - K = 4I - (4I - T_{n\times n})$

$$R = M^{-1}K = I - \frac{1}{4}T_{n\times n}$$

$\Rightarrow$ evals of R are $1 - \frac{1}{4}(\lambda_i + \lambda_j)$

λ_i are evals of T_n

$$\lambda_i = 2\left(1 - \cos\frac{i\pi}{n+1}\right)$$

$$\Rightarrow \rho(R) = 1 - \frac{1}{4} 2 \cdot \lambda_{min}$$

$$= 1 - \frac{1}{2}\lambda_{min}$$

$$= 1 - \left(1 - \cos\frac{\pi}{n+1}\right)$$

$$= \cos\frac{\pi}{n+1} \approx 1 - \frac{\pi^2}{2(n+1)^2} \quad \text{when } n \gg 1$$

$\rho(R)$ gets closer to 1 as n grows

$\Rightarrow$ slower convergence

$$\rho(R) = 1 - x \qquad x \ll 1$$

$$\rho(R)^m = (1-x)^m = (1-x)^{\frac{1}{x}\cdot mx}$$

$$\approx e^{-mx}$$

for $e^{-mx} = e^{-1} \Rightarrow m = \frac{1}{x}$

for Jacobi $\frac{1}{x} = \frac{2(n+1)^2}{\pi^2}$ for $n \gg 1$

$$= O(n^2) = O(N)$$

Cost to reduce error by any constant
also proportional to cost of N steps

cost = # iteration · # flops per iteration
$= O(N) \cdot O(N)$
$= O(N^2)$

Typical behavior: slower convergence
for larger problems (except multigrid!)

GS: Assuming updates in red/black order
$\rho(R_{GS}) = (\rho(R_J))^2$
$\Rightarrow$ 1 step of GS same as
2 steps of J
only 2x faster

SOR(w): Again with Red-Black ordering
optimal ω, much faster
$\rho(R_{SOR(\omega_{opt})}) \cong 1 - \frac{2\pi}{n}$
$\Rightarrow O(n) = O(N^{1/2})$ steps to converge
$\Rightarrow$ cost $= O(N^{3/2})$ to converge

Next: Present (and prove some of)
general convergence theory
for J, GS, SOR(w)
more details for Poisson

Thm 1: If A strictly row diagonally dominant

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}|$$

then J and GS converge

GS at least as fast as Jacobi

$$\| R_{GS} \|_\infty \le \| R_J \|_\infty < 1$$

Proof: just for J (see Thm 6.2 in text)

Splitting: $A = D - (D - A)$

$$R_J = D^{-1}(D-A) = I - D^{-1}A$$

$$\| R_J \|_\infty = \max_j \sum_i |R(j,i)|$$

$$= \left| 1 - \frac{A(j,j)}{A(j,j)} \right| + \sum_{i \neq j} \left| \frac{A(j,i)}{A(j,j)} \right| \quad \text{for some } j$$

$$= 0 + \frac{1}{|A(j,j)|} \sum_{i \neq j} |A(j,i)| < 1$$

by strict diagonal dominance

$\Rightarrow \| R_J \|_\infty < 1 \Rightarrow$ converges

2D Poisson: not strictly row diag. dominant

because most rows are $[-1 \;\; -1 \;\; 4 \;\; -1 \;\; -1]$

Def: If $|A(j,j)| \ge \sum_{i \neq j} |A(j,i)|$ for all j

with strict inequality at least once

$\Rightarrow$ A weakly row diagonally dominant

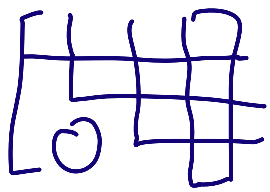
Not enough by itself for Jacobi to converge

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \Rightarrow R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow R^5 = R \Rightarrow R^i$ does not converge to 0

Need one more property of A, based on sparsity!

Def: A matrix is irreducible if there is no permutation P such that

$PAP^T =$  is block triangular

Equivalently, the directed graph corresponding to A ($A(i,j) \neq 0 \iff$ edge from i to j) is strongly connected: i.e. path from each i to each j

Model Problem is irreducible because graph is 1D, 2D, 3D, ... mesh

Model Problem also weakly diagonally dominant

Thm: If A is weakly row diagonally dominant and irreducible then $\rho(R_{GS}) < \rho(R_J) < 1$ so GS and J both converge, GS faster

(Thm 6.3 in text)

Thm: A s.p.d. $\Rightarrow$ SOR(w) converges iff $0 < w < 2$

In particular SOR(1) = GS converges

(Thm 6.5 text)

Convergence of SOR(w) for 2D Poisson

One more graph theory property:

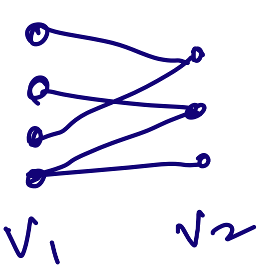
Def: A matrix has "Property A" if there is a permutation P such that

$$P A P^T = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$ s.t. A_{11} and A_{22} are diagonal

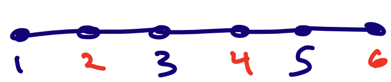
Graph Theory version: A is "bipartite" if can partition nodes (rows + cols)

$$V = V_1 \cup V_2$$

such that all edges in graph go from V_1 to V_2 or V_2 to V_1, none from V_1 to V_1 or V_2 to V_2



for 1D Poisson odd vertices $= V_1$
even vertices $= V_2$



for 2D Poisson: odd = black = V_1
even = red = V_2

same idea for 3D Poisson

Thm: Suppose A has "Property A" and we do SOR(w) updating all vertices in V_1 before V_2. Then evals μ of R_J and evals λ of $R_{SOR(w)}$ are related by

$$(*) \quad (\lambda + w - 1)^2 = \lambda w^2 \mu^2$$

If $w=1$, so SOR(1) = GS then $\lambda = \mu^2$

$$\Rightarrow \rho(R_{SOR(1)}) = \rho(R_{GS}) = (\rho(R_J))^2$$

$\Rightarrow$ GS converges twice as fast as J

proof: number vertices in V_1 before V_2

$$\Rightarrow A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right], \quad A_{ii} \text{ diagonal}$$

for 2D Poisson $A = 4I + \left[\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & 0 \end{array}\right]$

$$= 4I + \left[\begin{array}{c|c} 0 & 0 \\ \hline A_{21} & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & 0 \end{array}\right]$$

$$= D - L' - U'$$

λ eval of $R_{SOR(w)} \Rightarrow$

$$0 = \det(\lambda I - R_{SOR(\omega)})$$
$$= \det(\lambda I - (I-\omega L)^{-1}((1-\omega)I + \omega U))$$
$$= \det((I-\omega L)(\quad " \quad))$$
$$= \det(\lambda I - \omega\lambda L - ((1-\omega)I - \omega U))$$
$$= \det((\lambda - 1 + \omega)I - \omega\lambda L - \omega U)$$
$$= \det\left(\sqrt{\lambda}\,\omega\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} I - \sqrt{\lambda} L - \frac{1}{\sqrt{\lambda}} U\right)\right)$$
$$= (\sqrt{\lambda}\,\omega)^n \det\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} I - \sqrt{\lambda} L - \frac{1}{\sqrt{\lambda}} U\right)$$

$$D = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} I \end{bmatrix}$$

$$D(\sqrt{\lambda} L)D^{-1} = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{\lambda} A_{21} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \sqrt{\lambda} I \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} = L$$

$$D\left(\frac{1}{\sqrt{\lambda}} U\right) D^{-1} = U$$

$$= c \cdot \det(D(\,"\,)D^{-1})$$
$$= c \cdot \det\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} I - \underbrace{L - U}_{=-R_J}\right)$$
$$= c \cdot \det\left(\underbrace{\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega}}_{} I - R_J\right)$$

to be zero, this is eval of R_J

$$\Rightarrow \frac{\lambda - 1 + \omega}{\sqrt{\lambda}\,\omega} = \mu, \text{ same as } (*) \text{ above}$$

QED

Since we know all evals μ of R_J for Poisson can choose ω to minimize max eval λ of $R_{SOR(\omega)}$

Thm Suppose A has "property A" and SOR(ω) updates V_1 before V_2, and $\mu = \rho(R_J) < 1$ so Jacobi converges

then $$\omega_{opt} = \frac{2}{1+\sqrt{1-\mu^2}}$$

$$\rho(R_{SOR(\omega_{opt})}) = \omega_{opt} - 1 = \frac{\mu^2}{(1+\sqrt{1-\mu^2})^2}$$

for 2D Poisson: $$\omega_{opt} = \frac{2}{1+\sin(\frac{\pi}{n+1})} \lessapprox 2$$

$$\rho(R_{SOR(\omega_{opt})}) = \frac{\cos^2(\frac{\pi}{n+1})}{(1+\sin(\frac{\pi}{n+1}))^2} \sim 1 - \frac{2\pi}{n+1}$$ if n large

$\Rightarrow$ #steps to converge $\sim \sqrt{\text{\#steps for Jacobi to converge or GS}}$

Thm 6.5: A s.p.d $\Rightarrow$

$\rho(R_{SOR(\omega)}) < 1$ if $0 < \omega < 2$

$\omega = 1 \Rightarrow \rho(R_{SOR(1)}) = \rho(R_{GS}) < 1$

pf: $R = R_{SOR(\omega)}$ $\quad A = D - L' - U' = D(I - L - U)$

$$A = M - K = \left(\frac{1}{\omega}D - L'\right) - \left(\left(\frac{1}{\omega} - 1\right)D + U'\right)$$

1) $Q = A^{-1}(2M - A)$ ... show $Re(\lambda_i(Q)) > 0 \ \forall i$

2) $R = (Q - I)(Q + I)^{-1}$ ... implies $|\lambda_i(R)| < 1$

1) $Qx = \lambda x \Rightarrow (2M - A)x = \lambda Ax$

$$x^*(2M - A)x = \lambda x^* Ax$$

conjugate transpose: $x^*(2M^* - A)x = \bar{\lambda} x^* Ax$

add, divide by 2: $x^*(M + M^* - A)x = Re(\lambda)(x^* Ax)$

$$Re(\lambda) = \frac{x^*(M + M^* - A)x}{x^* Ax} = \frac{x^*\left(\frac{2}{\omega}D - L' - U' - (D - L' - U')\right)x}{x^* Ax}$$

$$= \frac{x^*\left(\left(\frac{2}{\omega} - 1\right)D\right)x^*}{x^* Ax}$$

$0 < \omega < 2 \Rightarrow \frac{2}{\omega} - 1 > 0 \Rightarrow$

$\left(\frac{2}{\omega} - 1\right)D$ s.p.d.

$$= \frac{pos}{pos} = pos$$

2) $(Q - I)(Q + I)^{-1} = (2A^{-1}M - 2I)(2A^{-1}M)^{-1}$

$$= I - M^{-1}A = I - M^{-1}(M - K)$$

$$= M^{-1}K = R$$

HWQ 4.5 (spectral mapping thm)

$$\lambda(R) = \frac{\lambda(Q) - 1}{\lambda(Q) + 1} \quad \text{for all evals}$$

$$|\lambda(R)|^2 = \frac{|\lambda(Q)-1|^2}{|\lambda(Q)+1|^2}$$

$$= \frac{(\operatorname{Re}\lambda(Q)-1)^2 + (\operatorname{Im}\lambda(Q))^2}{(\operatorname{Re}\lambda(Q)+1)^2 + (\operatorname{Im}\lambda(Q))^2}$$

since $\operatorname{Re}\lambda(Q) > 0$

< 1 QED