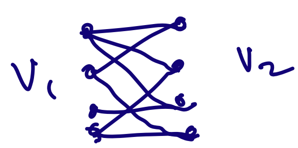
Welcome back to Ma 221! Lecture 37, Nov 20

Convergence of SOR(w) for 2D Poisson

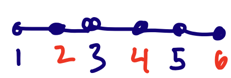
Def: A matrix has "Property A" if there is a permutation P such that

$$PAP^T = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \quad A_{11}, A_{22} \text{ diagonal}$$

Same Def via graph theory: A is "bipartite" if we can partition nodes (rows + cols) such that $V = V_1 \cup V_2$: all edges go from V_1 to V_2 (or V_2 to V_1) not V_1 to V_1 or V_2 to V_2 (ignore diagonal)

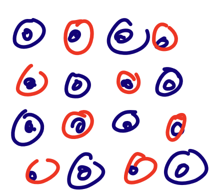


for 1D Poisson



even vertices $= V_2$
odd vertices $= V_1$

for 2D Poisson



$i+j$ (i,j) odd $= V_1$
$i+j$ (i,j) even $= V_2$

same idea for 3D

Thm: Suppose A has property A and we do SOR(ω) updating V_1 before V_2. Then evals μ of R_J and evals λ of $R_{SOR(\omega)}$ are related by

(*) $$(\lambda+\omega-1)^2 = \lambda\omega^2\mu^2$$

If $\omega=1$ so $SOR(1)=GS$ then $\lambda=\mu^2$

$\Rightarrow \rho(R_{SOR(1)}) = \rho(R_{GS}) = (\rho(R_J))^2$

$\Rightarrow$ GS converges twice as fast as Jacobi

proof: number V_1 before V_2

$$\Rightarrow A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{22} & A_{22} \end{array}\right] \quad A_{ii} \text{ diagonal}$$

for 2D Poisson $A = 4I + \left[\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & 0 \end{array}\right]$

$$= 4I + \left[\begin{array}{c|c} 0 & 0 \\ \hline A_{21} & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & 0 \end{array}\right]$$

$$= D - L' - U' = D(I-L-U)$$

λ is eval of $R_{SOR(\omega)} \Rightarrow$

$0 = \det(\lambda I - R_{SOR(\omega)})$

$= \det(\lambda I - (I-\omega L)^{-1}((1-\omega)I + \omega U))$

$= \det((I-\omega L) \quad (\quad " \quad))$ since $\det(I-\omega L)=1$

$= \det(\lambda I - \omega\lambda L - (1-\omega)I - \omega U)$

$$= \det\left((\lambda-1+\omega)I - \omega\lambda L - \omega U\right)$$

$$= \det\left(\sqrt{\lambda}\,\omega\left(\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega}\right)I - \sqrt{\lambda}L - \frac{1}{\sqrt{\lambda}}U\right)\right)$$

$$= (\sqrt{\lambda}\,\omega)^n \det\left(\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega}\right)I - \sqrt{\lambda}L - \frac{1}{\sqrt{\lambda}}U\right)$$

$$D = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\lambda}}I \end{bmatrix} \qquad D(\sqrt{\lambda}L)D^{-1} = L$$

$$D\left(\frac{1}{\sqrt{\lambda}}U\right)D^{-1} = U$$

$$= (\sqrt{\lambda}\,\omega)^n \det\left(D\,(\,''\,)\,D^{-1}\right)$$

$$= (\sqrt{\lambda}\,\omega)^n \det\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} I - L - U\right)$$

$$= (\sqrt{\lambda}\,\omega)^n \det\left(\frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} I - R_J\right)$$

$$\Rightarrow \forall \mu \in \text{eval}(R_J) \quad \frac{\lambda-1+\omega}{\sqrt{\lambda}\,\omega} = \mu$$

λ eval of $R_{SOR(\omega)}$ QED

Since we know all μ of R_J can choose ω to minimize $\rho(R_{SOR(\omega)})$

Thm: Suppose A has "property A" and SOR(ω) updates V_1 before V_2 and $\mu = \rho(R_J) < 1$ then $\omega_{opt} = \dfrac{2}{1+\sqrt{1-\mu^2}}$

$$\rho(R_{SOR(\omega_{opt})}) = \omega_{opt} - 1 = \frac{\mu^2}{\left(1+\sqrt{1-\mu^2}\right)^2}$$

for 2D Poisson: $\omega_{opt} = \dfrac{2}{1+\sin\left(\frac{\pi}{n+1}\right)} \lessapprox 2$

$$\rho(R_{SOR(w_{opt})}) = \frac{\cos^2\left(\frac{\pi}{n+1}\right)}{\left(1+\sin\left(\frac{\pi}{n+1}\right)\right)^2}$$

$$\approx 1 - \frac{2\pi}{n+1} \quad \text{for large } n$$

$\Rightarrow$ # steps to converge $\sim O\left(\frac{n+1}{2\pi}\right)$

$\sim \sqrt{\text{\# steps for Jacobi or GS}}$

(show Multigrid slides)