Welcome back to Ma221! Lecture 36, Nov 17

Splitting Methods

Def: Splitting of $A = M - K$, M nonsingular

$$Ax = b \iff Mx = Kx + b$$

solve iteratively by $M x_{i+1} = K x_i + b$

given x_0

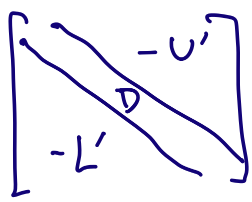
$$(*) \quad x_{i+1} = M^{-1} K x_i + M^{-1} b = R x_i + c$$

Def spectral radius $\rho(R) = \max_{\lambda \text{ eval of } R} |\lambda|$

Thm: $(*)$ converges $\forall x_0$ iff $\rho(R) < 1$

Describe: Jacobi, Gauss-Seidel (GS)

Successive Overrelaxation (SOR)

$A =$ 

$= D - L' - U' = D(I - L - U)$

Jacobi as a splitting $A = M - K = D - (L' + U')$

$$D x_{i+1} = (L' + U') x_i + b$$

$$R_J = M^{-1} K = D^{-1}(L' + U') = L + U$$

GS: $A = (D - L') - U'$
$\quad = M - K$

$$R_{GS} = M^{-1}K = (D - L')^{-1} U'$$
$$= (D(I - L))^{-1} U'$$
$$= (I - L)^{-1} U$$

SOR(w) $x^{SOR}_{w,i+1}(j) = (1-w)x_i(j) + w\, x_i^{GS}(j)$ depends on w

$w = 1 \Rightarrow SOR(1) = GS$
$w > 1 \Rightarrow$ "over relaxation"

Loop: for $j = 1:n$

$$x_{i+1}(j) = (1-w)x_i(j) + w \cdot \left(b_j - \sum_{k<j} A_{jk} x_{i+1}(k) - \sum_{k>j} A_{jk} x_i(k)\right) / A_{jj}$$

Splitting: Multiply by A_{jj}

$$(D - w \cdot L')x_{i+1} = D(1-w)x_i + wU'x_i + wb$$

divide by w

$$(D/w - L')x_{i+1} = \left(\left(\tfrac{1}{w} - 1\right)D + U'\right)x_i + b$$
$$M \quad x_{i+1} = K \quad x_i + b$$

$$A = (D/w - L') - (D/w - D + U')$$
$$= M - K$$

$$R_{SOR(w)} = (D/w - L')^{-1}\left(\left(\tfrac{1}{w}-1\right)D + U'\right)$$
$$= (I - wL)^{-1}\left(\left(1-\tfrac{1}{w}\right)I + wU\right)$$

For 2D Poisson

for all Red (j,k) ($j+k$ even)

$$V_{i+1}(j,k) = (1-w)V_i(j,k) + w\cdot \underbrace{(V_i(j-1,k) + V_i(j+1,k) + V_i(j,k-1) + V_i(j,k+1) + h^2F(j,k))/4}_{\text{old data}}$$

for all Black (j,k) ($j+k$ odd)

$$V_{i+1}(j,k) = (1-w)V_i(j,k) + w\cdot \underbrace{(V_{i+1}(j-1,k) + V_{i+1}(j+1,k) + V_{i+1}(j,k-1) + V_{i+1}(j,k+1) + h^2F(j,k))/4}_{\text{updated data}}$$

Convergence of Splitting Methods

Jacobi for 2D Poisson

$$T_{n\times n} = M - K = 4I - (4I - T_{n\times n})$$
$$\Rightarrow R = M^{-1}K = I - \tfrac{1}{4}T_{n\times n}$$

$\Rightarrow$ evals of R are $1-(\lambda_i+\lambda_j)/4$

where λ_i are evals of T_n

$$\lambda_i = 2(1-\cos\frac{i\pi}{n+1})$$

$$\Rightarrow \rho(R) = 1 - \frac{\lambda_{min}}{2}$$

$$= 1-(1-\cos\frac{\pi}{n+1})$$

$$= \cos\frac{\pi}{n+1}$$

$$\approx 1 - \frac{\pi}{2(n+1)^2} \quad \text{for large } n$$

$\Rightarrow \rho(R)$ gets closer to 1 as n grows

$\Rightarrow$ slower convergence

Error after m steps smaller by $(\rho(R))^m$

$$\rho(R) = 1-x$$

$$(\rho(R))^m = (1-x)^m = (1-x)^{\frac{1}{x}mx}$$

$$\approx e^{-mx} \quad \text{for } x \ll 1$$

for $e^{-mx} = e^{-1} \Rightarrow m = \frac{1}{x}$

for Jacobi $\frac{1}{x} = \frac{2(n+1)^2}{\pi^2}$ for $n \gg 1$

$$= O(n^2)$$

$$= O(N) \quad N = \text{dimension}$$

$$= O(\text{cond}(T_{n\times n}))$$

iterations to reduce error by any constant factor

$$= O(N)$$

cost = # iterations $\cdot$ # flops_per_iteration

$= O(N) \cdot O(N)$
$= O(N^2)$

Typical: slower convergence for larger problems (but not multigrid!)

GS: Assuming Red-Black ordering

$\rho(R_{GS}) = (\rho(R_J))^2$

$\Rightarrow$ 1 step of GS same as 2 steps of Jacobi

only constant factor faster

SOR(ω): Again with Red-Black, optimal ω much faster: $\rho(R_{SOR(\omega_{opt})}) \approx 1 - \frac{2\pi}{n}$

$\Rightarrow O(n) = O(N^{1/2})$ steps to converge

$\Rightarrow$ cost $= O(N^{3/2})$

Thm 1: If A strictly row diagonally dominant

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}|$$

then both Jacobi, GS converge:

GS at least as fast as Jacobi:

$$\|R_{GS}\|_\infty \leq \|R_J\|_\infty < 1$$

Proof: just for Jacobi (Thm 6.2 for GS)

Split: $A = D - (D-A)$

$$R = D^{-1}(D-A) = I - D^{-1}A$$

$$\|R\|_\infty = \max_j \sum_i |R(j,i)|$$

$$= \left|1 - \frac{A_{jj}}{A_{jj}}\right| + \sum_{i \neq j} \frac{|A_{ji}|}{|A_{jj}|}$$

$$= \frac{1}{|A_{jj}|} \sum_{i \neq j} |A_{ji}| < 1$$

by diagonal dominance

2D Poisson has rows: $[-1 \quad -1 \quad 4 \quad -1 \quad -1]$

not "strictly" diagonally dominant

Def A weakly row diag dom if

$|A_{jj}| \geq \sum_{i \neq j} |A_{ji}|$, strict inequality at least once

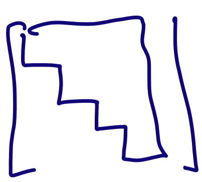
Not enough for Jacobi to converge

Ex $A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \Rightarrow R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow R^5 = R \Rightarrow R^i$ does not converge

Need one more property of A

Def: A irreducible if there is no permutation P such that

$PAP^T =$  is block triangular

Equivalently: the directed graph corresponding to A ($A(i,j) \neq 0 \Leftrightarrow$ edge (i,j))

is strongly connected

i.e. $\exists$ path from any i to any j

What about Poisson? meshes are strongly connected

Thm: If A weakly row diag dominant and irreducible then

$\rho(R_{GS}) < \rho(R_J) < 1$

(Thm 6.3)

Thm: A s.p.d. $\Rightarrow$ SOR(w) converges iff $0 < w < 2$

in particular SOR(1) = GS converges

(Thm 6.5)