Welcome back to Ma221! Lecture 33, Nov 8

Model Problem: Poisson Equation

1D: Discretizing ODE with Dirichlet boundary conditions

$$-\frac{d^2}{dx^2} v(x) = f(x) \quad \text{on } [0,1]$$

$$v(0) = v(1) = 0$$

Recall Lecture 14-15: discretize to get

$$T_N \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} = T_N v = h^2 \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} = h^2 f \qquad h = \frac{1}{N+1}$$

$$T_N = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}$$ stencil 

Evals and evecs of T_N

Lemma: $T_N \cdot z_j = \lambda_j \cdot z_j$ where $\|z_j\|_2 = 1$

$$\lambda_j = 2\left(1 - \cos \frac{\pi j}{N+1}\right)$$

$$z_j(k) = \sqrt{\frac{2}{N+1}} \sin\left(j \cdot k \cdot \pi / (N+1)\right)$$

proof: trig (HW Q6.1)

Corollary: Z : $Z_{jk} = \sqrt{\frac{2}{N+1}} \sin\left(j \cdot k \cdot \pi/(N+1)\right)$ is orthogonal

$Z = \text{Imag}(\text{FFT})$

Evals range from $\lambda_j \sim \left(\frac{\pi \cdot j}{N+1}\right)^2$ for small j

and for large N, up to $\lambda_N \sim 4$

$$\Rightarrow \text{cond}\# = \frac{\lambda_N}{\lambda_1} = \left(\frac{2(N+1)}{\pi}\right)^2 = \left(\frac{2}{\pi}\right)^2 h^{-2}$$

2D Poisson with Dirichlet Boundary Conds.

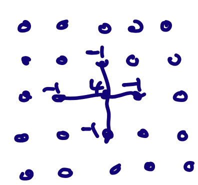
$$-\frac{\partial^2 v(x,y)}{\partial x^2} - \frac{\partial^2 v(x,y)}{\partial y^2} = f(x,y) \text{ on } [0,1]^2$$

with $v(x,y) = 0$ on boundary of $\square$

discretize as before $v_{ij} = v(i \cdot h, j \cdot h)$, $h = \frac{1}{N+1}$

$$(*)\quad 4 v_{ij} - \overset{\text{above}}{v_{i-1,j}} - \overset{\text{below}}{v_{i+1,j}} - \overset{\text{right}}{v_{i,j+1}} - \overset{\text{left}}{v_{i,j-1}} = h^2 f_{ij}$$

2D stencil



$V = N \times N$ matrix of unknowns

$$(*)\quad \begin{cases} 2v_{ij} - v_{i-1,j} - v_{i+1,j} = (T_N V)_{ij} \\ 2v_{ij} - v_{i,j+1} - v_{i,j-1} = (V T_N)_{ij} \end{cases}$$

$$(*)\quad T_N V + V T_N = h^2 F$$

N^2 eqns in N^2 unknowns V

Sylvester Eqn (Q4.6)

Evals and Evecs of 2D Poisson:

$$T_N V + V T_N = \lambda V$$

$\lambda =$ eval, $V =$ "evec"

Suppose $V = z_i \cdot z_j^T$ where $T_N z_i = \lambda_i z_i$

$$T_N V + V T_N = (T_N \cdot z_i) z_j^T + z_i (z_j^T T_N)$$
$$= \lambda_i (z_i z_j^T) + z_i (\lambda_j z_j^T)$$
$$= (\lambda_i + \lambda_j)(z_i z_j^T)$$
$$= (\lambda_i + \lambda_j) \cdot V$$

$V =$ "evec", eval $= \lambda_i + \lambda_j$

$\exists$ eval, evec for all pairs (i,j)

Express V as vector to generalize to 3D and higher:

3x3 case: write V columnwise, left to right

$$v = [V_{11}, V_{21}, V_{31}, V_{12}, \ldots, V_{33}]^T$$

$$T_{N\times N} = \left[\begin{array}{ccc|ccc|ccc} 4 & -1 & & -1 & & & & & \\ -1 & 4 & -1 & & -1 & & & & \\ & -1 & 4 & & & -1 & & & \\ \hline -1 & & & 4 & -1 & & -1 & & \\ & -1 & & -1 & 4 & -1 & & -1 & \\ & & -1 & & -1 & 4 & & & -1 \\ \hline & & & -1 & & & 4 & -1 & \\ & & & & -1 & & -1 & 4 & -1 \\ & & & & & -1 & & -1 & 4 \end{array}\right]$$

multiplies $v_{i,j\pm1}$

multiplies $v_{i\pm1,j}$

$$= \left[\begin{array}{c|c|c} T_N + 2I_N & -I_N & \\ \hline -I_N & T_N + 2I_N & -I_N \\ \hline & -I_N & T_N + 2I_N \end{array}\right]$$

Generalize to larger N, higher dimensions, using Kronecker Product

Def: $X^{m \times n}$ then $\text{vec}(X)$ defined as $m \cdot n \times 1$ vector gotten by stacking columns of X on top of one another, left to right

Matlab: reshape(X, m·n, 1)

Def: Let $A^{m \times n}$, $B^{p \times q}$

Then $A \otimes B$ is $m \cdot p \times n \cdot q$

$$\begin{bmatrix} A_{11} \cdot B & A_{12} \cdot B & \cdots & A_{1n} \cdot B \\ A_{21} \cdot B & A_{22} \cdot B & & \\ \vdots & & \ddots & \\ A_{m1} \cdot B & & & A_{mn} \cdot B \end{bmatrix}$$

is Kronecker product of A and B

Matlab: kron(A, B)

Lemma: Given $A^{m \times m}$, $B^{n \times n}$, $X^{m \times n}$

1) $\text{vec}(A \cdot X) = (I_n \otimes A) \cdot \text{vec}(X)$

2) $\text{vec}(X \cdot B) = (B^T \otimes I_m)\,\text{vec}(X)$

3) 2D Poisson $T_N V + V T_N = F$ same as

$$(I_N \otimes T_N + \underbrace{T_N}_{= T_N^T} \otimes I_N)\,\text{vec}(V) = \text{vec}(F)$$

proof: 1) $I_N \otimes A = \begin{bmatrix} A & & 0 \\ & A & \\ & \ddots & \\ 0 & & A \end{bmatrix} = diag(\underbrace{A, \dots, A}_{n \text{ times}})$

$$(I_N \otimes A) \text{vec} X = \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix} \begin{bmatrix} X(:,1) \\ X(:,2) \\ \vdots \\ X(:,n) \end{bmatrix} = \begin{bmatrix} A \cdot X(:,1) \\ \vdots \\ A \cdot X(:,n) \end{bmatrix}$$

$$= \text{vec}(A \cdot X)$$

2) similar (HW Q6.4)

3) Apply 1) to $T_N \cdot V$ and 2) to $V \cdot T_N$

$$I_N \otimes T_N + T_N \otimes I_n = \begin{bmatrix} T_N & & \\ & \ddots & \\ & & T_N \end{bmatrix} + \begin{bmatrix} 2 \cdot I_n & -I_N & & 0 \\ -I_N & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} T_N + 2I_N & -I_N & & \\ -I_N & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

Lemma (HW Q6.4)

1) Assume $A \cdot C$ and $B \cdot D$ well defined

$$(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$$

2) A and B invertible $\Rightarrow$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

3) $(A \otimes B)^T = A^T \otimes B^T$

Prop: Let $T = Z \Lambda Z^T$ be eigen decomp of T
$N \times N$ symmetric

Then eigendecomp of
$I_N \otimes T + T \otimes I_N$ is

$$(*) \quad (Z \otimes Z)\underbrace{(I_N \otimes \Lambda + \Lambda \otimes I_N)}_{\text{diagonal with}}(Z \otimes Z)^T$$

$((i-1)N+j)^{th}$ diagonal $= \lambda_i + \lambda_j$

$(Z \otimes Z)$ orthogonal with
$((i-1)\cdot N+j)^{th}$ column $z_i \otimes z_j$

proof: multiply out $(*)$:

$$(Z \cdot I_N \otimes Z \cdot \Lambda + Z \cdot \Lambda \otimes Z \cdot I_N)(Z \otimes Z)^T$$
$$= (\quad \text{"} \quad)(Z^T \otimes Z^T)$$
$$= (Z \cdot I_N \cdot Z^T \otimes Z \cdot \Lambda \cdot Z^T + Z \cdot \Lambda \cdot Z^T \otimes Z \cdot I_N \cdot Z^T)$$
$$= (I_N \otimes T_N + T_N \otimes I_N)$$

Poisson equation in 3 (or any) D

$$T_{N \times N \times N} = (T_N \otimes I_N \otimes I_N) + (I_N \otimes T_N \otimes I_N) + (I_N \otimes I_N \otimes T_N)$$

with eigenvalue matrix

$$(\Lambda_N \otimes I_N \otimes I_N) + (I_N \otimes \Lambda_N \otimes I_N) + (I_N \otimes I_N \otimes \Lambda_N)$$

N^3 evals $\lambda_i + \lambda_s + \lambda_j$ for all (i,j,k)

eigenvector matrix: $Z \otimes Z \otimes Z$

Solving Poisson with FFT

direct method, not iterative

Start with 2D Poisson

$$T_N \cdot V + V \cdot T_N = F \quad , T_N = Z \cdot \Lambda \cdot Z^T$$

$$Z^T \left(Z \cdot \Lambda \cdot Z^T \cdot V + V \cdot Z \cdot \Lambda \cdot Z^T = F \right) Z$$

$$\Lambda \cdot (Z^T V Z) + (Z^T V Z) \cdot \Lambda = Z^T F Z$$

$$\Lambda \cdot V' + V' \cdot \Lambda = F'$$

diagonal Sylvester eqn

$$(\Lambda \cdot V')_{ij} + (V' \cdot \Lambda)_{ij} = (F')_{ij}$$

$$\lambda_i V'_{ij} + V'_{ij} \lambda_j = F'_{ij}$$

$$(*) \qquad V'_{ij} = F'_{ij} / (\lambda_i + \lambda_j)$$

1) compute $F' = Z^T F Z$
2) solve $(*)$ for V' costs $O(n^2)$
3) compute $V = Z V' Z^T$

if Steps 1, 2 used matmul, would cost $O(n^3)$ but Z related to FFT, costs $O(n^2 \log n)$

cost $= O(n^2 \log n)$

$$FFT(i,j) = e^{2\pi\sqrt{-1}\, i \cdot j / N}$$

Extends to higher dim Poisson using Kronecker product