Welcome back to Ma221! Lecture 29, Oct 30

Symmetric Eigenproblem, for real $A = A^T$

$$A = Q \Lambda Q^T$$

$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \quad \lambda_1 \ge \cdots \ge \lambda_n$

$Q = [q_1, \ldots, q_n]$

all extends to svd of B via $\left[\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right]$

Def Rayleigh Quotient $\rho(u, A) = \dfrac{u^T A u}{u^T u}$

$u = Qb \Rightarrow$

$$\rho(u, A) = \sum_{i=1}^n b_i^2 \lambda_i \Big/ \sum_{i=1}^n b_i^2$$

$$= \sum_{i=1}^n w_i \lambda_i \qquad \sum_{i=1}^n w_i = 1, \quad w_i \ge 0$$

Courant-Fischer Minimax Thm

R^j = j-dimensional subspace of $\mathbb{R}^n$

S^{n-j+1} = $n-j+1$ dim. " " "

$$\lambda_j = \max_{R^j} \min_{\substack{r \in R^j \\ r \ne 0}} \rho(r, A) = \min_{S^{n-j+1}} \max_{\substack{s \in S^{n-j+1} \\ s \ne 0}} \rho(s, A)$$

max over R^j attained by $\text{span}(q_1, \ldots, q_j)$

min over $r \in R^j$ " " q_j

min over S^{n-j+1} " " span $(q_j, q_{j+1} \cdots q_n)$

max over $s \in S^{n-j+1}$ " " q_j

proof: Given any R^j and S^{n-j+1}

dimensions add up to $n+1$

$\Rightarrow$ intersect in some nonzero x_{RS}

$$\lambda_j \overset{\text{goal}}{\leq} \min_{\substack{r \in R^j \\ r \neq 0}} \rho(r, A) \leq \rho(x_{RS}, A) \leq \max_{\substack{s \in S^{n-j+1} \\ s \neq 0}} \rho(s, A) \overset{\text{goal}}{\leq} \lambda_j$$

Let $\underset{\dim R' = j}{R'}$ maximize $\min_{\substack{r \in R^j \\ r \neq 0}} \rho(r, A)$

Let $\underset{\dim S' = n-j+1}{S'}$ minimize $\max_{\substack{s \in S^{n-j+1} \\ s \neq 0}} \rho(s, A)$

$$\lambda_j \leq \max_{R^j} \min_{\substack{r \in R^j \\ r \neq 0}} \rho(r, A) = \min_{\substack{r \in R' \\ r \neq 0}} \rho(r, A) \leq \rho(x_{R'S'}, A) \leq \max_{\substack{s \in S' \\ s \neq 0}} \rho(s, A) = \min_{S^{n-j+1}} \max_{\substack{s \in S^{n-j+1} \\ s \neq 0}} \rho(s, A) \leq \lambda_j$$

If we choose $R^j = \text{span}(q_1, \ldots, q_j)$

$r = q_1 \Rightarrow \min_{\substack{r \in R^j \\ r \neq 0}} \rho(r, A) = \rho(q_j, A) = \lambda_j$

If we choose $S^{n-j+1} = \text{span}(q_j, \ldots, q_n)$

$s = q_j \Rightarrow \max_{\substack{s \in S^{n-j+1} \\ s \neq 0}} \rho(s, A) = \rho(q_j, A) = \lambda_j$

all above inequalities are equalities QED

Weyl's Thm: $A = A^T$ with evals $\lambda_1 \geq \cdots \geq \lambda_n$
and $E = E^T$ where
$A+E$ has evals $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$
then $|\lambda_i - \mu_i| \leq \|E\|_2$ for $1 \leq i \leq n$

Corollary for SVD
If A general, sing vals $\sigma_1 \geq \cdots \geq \sigma_n$
If $A+E$ " sing vals $\tau_1 \geq \cdots \geq \tau_n$
$|\tau_i - \sigma_i| \leq \|E\|_2$ for $1 \leq i \leq n$

Proof of Weyl:

$$\mu_j = \min_{S^{n-j+1}} \max_{\substack{s \in S^{n-j+1} \\ s \neq 0}} \frac{s^T(A+E)s}{s^Ts}$$

$$= \min \max \frac{s^TAs}{s^Ts} + \frac{s^TEs}{s^Ts}$$

$$\leq \min \max \frac{s^TAs}{s^Ts} + \|E\|_2$$

$$= \lambda_j + \|E\|_2$$

swap μ and λ to get $\lambda_j \leq \mu_j + \|E\|_2$

QED

Def: Inertia(A) = (# neg evals of A,
zero evals of A,
pos evals of A)

Sylvester's Thm: $A = A^T$, X nonsingular
$\Rightarrow$ inertia(A) = inertia(X^TAX)

Fact: Suppose we factor $A = LDL^T$ or $PLDL^TP^T$
(Gauss elim with symmetric or no pivoting)
inertia(A) = inertia(D)
$= (\# D_{ii} < 0, \# D_{ii} = 0, \# D_{ii} > 0)$

Factor $A - xI = L' \cdot D' \cdot L'^T$
$\# D'_{ii} < 0$ = # evals of $A - xI < 0$
= # evals of $A < x$

Factor $A - yI$, $y > x$, $A - yI = L'' \cdot D'' \cdot L''^T$
$\# D''_{ii} < 0$ = # evals of $A < y$

$\Rightarrow$ # evals of A in $[x, y) = \# D''_{ii} < 0 - \# D'_{ii} < 0$

$\Rightarrow$ count # evals of A in any interval

next compute # evals $< \frac{x+y}{2}$ etc
keep bisecting nonempty intervals
until narrow enough

Cheaper when only want top k evals
or evals $\lambda_3, \lambda_4, \lambda_5$ etc

Can make it cost $O(n)$ by first
reducing $A = Q^T T Q$, $Q^T Q = I$
T tridiagonal
(in LAPACK, xsyevx)

proof of Sylvester's Thm: by contradiction

Suppose # evals of $A<0$ is m

and # evals of $X^TAX<0$ is $m'<m$

N = m-dimensional subspace for m negative evals of A

$\Rightarrow 0 \neq x \in N \Rightarrow x^TAx<0$

P = $n-m'$ dimensional subspace for $n-m'$ nonnegative evals of X^TAX

$\Rightarrow 0 \neq x \in P \Rightarrow x^T(X^TAX)x \geq 0$

$= (Xx)^TA(Xx) \geq 0$, $Xx = y$

$y^TAy \geq 0$

$\dim(XP) + \dim(N) > n$

$n-m' + m > n \quad m'<m$

$\Rightarrow XP$ and N intersect in $z \neq 0$

$\Rightarrow z^TAz \geq 0$ and $z^TAz<0$

contradiction QED

Perturbation Theory for Evecs

Thm: $A = Q\Lambda Q^T$

$A+E = Q'\Lambda'Q'^T \qquad \Lambda' = \mathrm{diag}(\lambda_1', \ldots, \lambda_n')$

$Q' = [q_1', \ldots q_n']$

Goal: bound angle $\angle(q_i, q_i') = \theta_i$

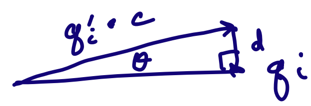
Def: $\mathrm{gap}(i,A) = \min_{j \neq i} |\lambda_j - \lambda_i|$

$$\underbrace{|.5 \sin(2\theta_i)|}_{\sim \theta_i \text{ for } \theta_i \ll 1} \leq \frac{\|E\|_2}{\mathrm{gap}(i,A)}$$

Worst Case: $A = \gamma I$, $\text{gap}(i, A) = 0 \quad \forall i$

proof of weaker result (full result in text)

evec of $A+E$ is $q_i + d \qquad d^T q_i = 0$



: bound $\|d\|_2 = \tan\theta$

$$q_i' = \frac{q_i + d}{\|q_i + d\|_2}$$

$$(A+E)(q_i + d) = \lambda_i' (q_i + d)$$

$$A q_i + Ad + E q_i + \underbrace{Ed}_{\text{ignore second order term}} = \lambda_i' q_i + \lambda_i' d$$

$$(A + E - \lambda_i' I)\, q_i = (\lambda_i' I - A) d$$

$$(\lambda_i I + E - \lambda_i' I) q_i = (\lambda_i' I - A) d \quad , \; d = \sum_{j \neq i} d_j q_j$$

$$2\|E\|_2 \geq \text{LHS} = \|((\lambda_i - \lambda_i') I + E) q_i\|_2 = \|\sum_{j \neq i} (\lambda_i' - \lambda_j) d_j q_j\|_2 = \text{RHS}$$

$\text{LHS} \leq 2\|E\|_2$ because $|\lambda_i - \lambda_i'| \leq \|E\|_2$ by Weyl

$$\text{RHS} \geq (\text{gap}(i, A) - \|E\|_2)\, \|d\|_2$$

$$\frac{2\|E\|_2}{\text{gap}} \approx \frac{2\|E\|_2}{\text{gap} - \|E\|} \geq \|d\|_2 = |\tan\theta| \sim |\theta| \quad \text{if } |\theta| \ll 1$$

assuming $\|E\| \ll \text{gap}$ QED