Welcome back to Ma 221! Lecture 24, Oct 18

Eigenproblems

Lemma: if $A x_i = \lambda_i x_i$ for $i = 1:n$

and $S = [x_1, \dots, x_n]$ nonsingular

$\Rightarrow A = S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} S^{-1}$

Conversely if $A = S \Lambda S^{-1}$, $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

then columns of S are evecs

proof: $A = S \Lambda S^{-1}$ iff $AS = S\Lambda$

iff $A\, S[:,i] = S[:,i] \cdot \lambda_i$ i.e.

$S[:,i]$ evec for eval λ_i

But we can't always diagonalize A:

may be mathematically impossible

(recall Jordan form)

may be unstable even if it exists

(when evals close together)

Recall Jordan Form: For any A

$\exists$ similarity S: $SAS^{-1} = J = \mathrm{diag}(J_1, \dots, J_k)$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

upto permuting order of J_i, unique.

J_i can be same or different eg $A = I$

only one right or left evec per λ_i:

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \text{can't diagonalize}$$

Why not compute Jordan Form?

Consider slightly perturbed 2x2 I

① $\begin{bmatrix} 1 & 0 \\ 0 & 1+e \end{bmatrix}$: $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(1+e, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

② $\begin{bmatrix} 1 & e \\ e & 1 \end{bmatrix}$: $\left(1+e, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \left(1-e, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

evecs rotate $45°$

③ $\begin{bmatrix} 1 & e \\ 0 & 1+e^2 \end{bmatrix}$: $\left(1 : \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(1+e^2, \begin{bmatrix} 1 \\ e \end{bmatrix}\right)$

evecs nearly parallel

④ $\begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix}$: $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ just one evec

⑤ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: $(1 : \text{anything}), (1, \text{anything})$

Jordan form very ill conditioned

Our goal: exact evals, evecs of

$$A+E, \quad \frac{\|E\|}{\|A\|} = O(\varepsilon)$$

Backward Stable approach to eigenproblem:

Only use orthogonal similarities

Recall Chap 3: Multiplying by multiple orthog. matrices is backward stable

$$fl\left(Q_k(\cdots(Q_2(Q_1 A)\cdots)\right) = \underset{\text{exact orthog}}{Q}(A+E) \qquad \frac{\|E\|}{\|A\|} = O(\varepsilon)$$

Apply twice to orthog. similarity

$$fl\left(Q_k(\cdots Q_2(Q_1 A Q_1^T)Q_2^T \cdots)Q_k^T\right) = Q(A+F)Q^T$$

$$QQ^T = I, \quad \frac{\|F\|}{\|A\|} = O(\varepsilon)$$

If we restrict to orthog similarities, how close to Jordan form can we get?

Thm: Schur Canonical Form:

Given any $n \times n$ A $\exists$ unitary Q

$QQ^H = I$ s.t. $Q^H A Q = T =$ upper triangular

evals of T are $T(i,i)$

Compute evecs of T just triangular solve.

$$\begin{array}{c} \\ i-1 \\ 1 \\ n-i \end{array}
\begin{array}{c} \begin{array}{ccc} i-1 & 1 & n-i \end{array} \\
\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T(i,i) & T_{23} \\ 0 & 0 & T_{33} \end{bmatrix} \end{array}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T(i,i) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T_{11}x_1 + T_{12}x_2 + T_{13}x_3 = T(i,i)\,x_1$$
$$T(i,i)x_2 + T_{23}x_3 = T(i,i)\,x_2$$
$$T_{33}\,x_3 = T(i,i)\,x_3$$

Suppose $T(i,i)$ unique

$$(\underbrace{T_{33} - T(i,i)\,I}_{\text{non zero diagonal}})x_3 = 0 \quad \Rightarrow \quad x_3 = 0$$

$$T(i,i)\,x_2 = T(i,i)\,x_2 \qquad x_2 = 1$$

$$(T_{11} - T(i,i)\,I)\,x_1 = -T_{12} \cdot x_2$$

triangular solve

What does Matlab do with

$[V, D] = \text{eig}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$? try it!

Proof that Schur form exists:

choose any λ, with evec x: $Ax = \lambda x$

$\|x\|_2 = 1$

let $Q = [x, Q']$ be unitary

$$Q^H A Q = \begin{bmatrix} x^H \\ Q'^H \end{bmatrix} A \, [x, Q']$$

$$= \begin{array}{c} \\ 1 \\ n-1 \end{array} \begin{array}{c} \begin{array}{cc} 1 & n-1 \end{array} \\ \left[\begin{array}{c|c} x^H A x & x^H A Q' \\ \hline Q'^H A x & Q'^H A Q' \end{array}\right] \end{array}$$

$$= \left[\begin{array}{c|c} x^H \lambda x & \text{''} \\ \hline Q'^H \lambda x & \text{''} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & Q'^H A Q' \end{array}\right]$$

induction on $Q'^H A Q' = U^H \underbrace{T}_{\text{triangular}} \overbrace{U}^{\text{unitary}}$

$$= \left[\begin{array}{c|c} \lambda & x^H A Q' \\ \hline 0 & U^H T U \end{array}\right]$$

$$= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U^H \end{array}\right] \left[\begin{array}{c|c} \lambda & x^* A Q' U^H \\ \hline 0 & T \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U \end{array}\right]$$

$$\Longrightarrow \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U \end{array}\right] Q^H}_{\text{unitary}} A \underbrace{Q \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U^H \end{array}\right]}_{\text{inverse}} = \underbrace{\left[\begin{array}{c|c} \lambda & x^H A Q' U^H \\ \hline 0 & T \end{array}\right]}_{\text{Schur form}}$$

What about real matrices with complex evals?

[if $A = A^T$, all evals real, see Chap 5]

Prefer real arithmetic!

- reduce # flops
- less memory
- make sure that all evals, evecs appear in complex conjugate pairs

$$\underset{\text{real}}{A}\ \underset{\text{complex}}{x} = \lambda x \iff A\bar{x} = \bar{\lambda}\bar{x}$$

Instead of $T =$ upper triangular
use $T =$ block triangular

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1k} \\ & T_{22} & & \\ & & \ddots & \\ & & & T_{kk} \end{bmatrix} \quad T_{ii} \text{ square}$$

$$\text{evals}(T) = \bigcup_{i=1}^{k} \text{evals}(T_{ii}) \quad (\text{HW } 4.1)$$

Goal: have all T_{ii} be 1×1 for real evals
or 2×2 for complex conjugate pairs $\lambda, \bar{\lambda}$

Thm: (Real Schur Canonical form)
Given any real A, $\exists$ real orthog Q
s.t. QAQ^T is block upper triangular
with 1×1 and 2×2 diag. blocks

Generalize to "invariant subspaces"

Def: $V = \text{span}\{x_1, \ldots, x_m\} = \text{span}(X)$
$X = [x_1 \ldots x_m]$ subspace of $\mathbb{R}^n$
V invariant if $A \cdot V = \text{span}(AX) \subseteq V$

Ex: $V = \text{span}\{x\} = \{\alpha x$ for all $\alpha \in \mathbb{R}\}$
where $Ax = \lambda x$
$AV = \{A(\alpha x)\ \forall \alpha \in \mathbb{R}\}$

$$= \{\alpha \lambda x, \forall \alpha\}$$

$$\subseteq V \quad , = V \quad \text{unless } \lambda = 0$$

Ex: $V = \text{span}\{x_1, \dots, x_k\} = \{\sum_{i=1}^{k} \alpha_i x_i \quad \forall \alpha_i \in \mathbb{R}\}$
where $Ax_i = \lambda_i x_i$

$$AV = \{A \sum \alpha_i x_i\} = \{\sum \alpha_i A x_i\}$$

$$= \{\sum \alpha_i \lambda_i x_i\} \subseteq V$$

Lemma: $V = \{x_1, \dots, x_n\}$ invariant
then $\exists B: AX = XB^{n \times n}$

the evals of B are evals of A

proof: existence of B follows from def of invariance

$Ax_i \in V \Rightarrow \exists$ scalars $B(1,i), B(2,i) \dots B(n,i)$

$$Ax_i = \sum_{j=1}^{n} x_j B(j,i) \quad \text{i.e. } AX = XB$$

$$By = \lambda y \Rightarrow A(Xy) = (AX)y = (XB)y = X(By) = \underbrace{(Xy)}_{\text{evec of } A}\lambda$$

Lemma: Let $V = \text{span}(X)$ be n-dimensional Invariant subspace of A,

$$AX = XB \quad , \quad X = QR$$

Let $[Q, Q']$ be square orthog matrix

$$[Q, Q']^T A [Q, Q'] = \begin{array}{cc} & n \\ n & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] \end{array}$$

$A_{11} = RBR^{-1}$ has same evals as B

proof: $$[Q, Q']^T A [Q, Q'] = \left[\begin{array}{c|c} Q^T A Q & Q^T A Q' \\ \hline Q'^T A Q & Q'^T A Q' \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{c|c} RBR^{-1} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

$$AQ = AXR^{-1} = XBR^{-1} = QRBR^{-1}$$

$$A_{11} = Q^T AQ = \underbrace{Q^T Q}\, RBR^{-1} = RBR^{-1}$$

$$A_{21} = \underbrace{Q' Q}_{0}\, RBR^{-1} = 0$$