Computing the eigenpairs of W_{21}^{+} with MRRR

Math 221, November 15, 2004.

Christof Vömel

voemel@eecs.berkeley.edu

UC Berkeley

The essence of the MRRR algorithm

Given an RRR for a set of eigenvalues:

FOR EACH eigenvalue with a large relative gap

- Compute eigenvalue to high rel. accuracy.
- Compute the FP vector (eigenvector).

FOR EACH of the remaining groups of eigenvalues

- Choose shift σ outside the group.
- Compute new RRR $L_+ D_+ L_+^T = LDL^T - \sigma I$.
- Refine the eigenvalues.

Wilkinson matrix W_{21}^{+}

```
function T = wilkin(n)
m=(n-1)/2;
T = diag(abs(-m:m))+diag(ones(2*m,1),1)
                          +diag(ones(2*m,1),-1);
```

$$W_{21}^{+} = \text{tridiag}\begin{pmatrix} & 1 & & 1 & & \ldots & & 1 & & 1 & \\ 10 & & 9 & & \ldots & 0 & \ldots & & 9 & & 10 \\ & 1 & & 1 & & \ldots & & 1 & & 1 & \end{pmatrix}$$

Eigenvalues of W_{21}^{+}

1	-1.125441522119984	12	6.000217522257097
2	0.253805817096679	13	6.000234031584167
3	0.947534367529293	14	7.003951798616375
4	1.789321352695081	15	7.003952209528675
5	2.130209219362507	16	8.038941115814273
6	2.961058884185726	17	8.038941122829025
7	3.043099292578824	18	9.210678647304919
8	3.996048201383624	19	9.210678647361332
9	4.004354023440857	20	10.746194182903322
10	4.999782477742902	21	10.746194182903393
11	5.000244425001912		

Compute root representation

All eigenvalues are in
$[-1.1254415288568682, 10.746194215443904]$

- Choose $\sigma = 10.746194215443904$
- Compute $LDL^T = T - \sigma I$.
- Compute eigenvalues of LDL^T.

Relatively isolated $\Leftrightarrow$ relgap > 10^{-3}

$\Leftrightarrow \approx$ less that 3 digits in common

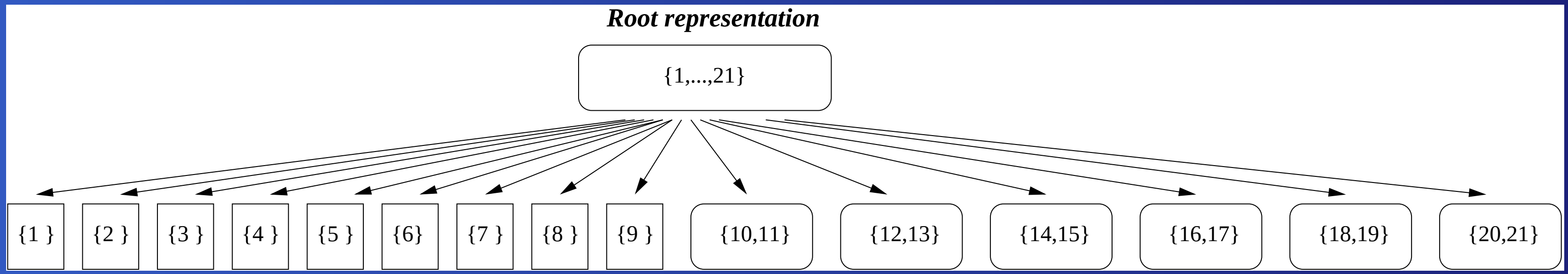
Eigenvalues of root representation

1	-11.8716357375639	12	-4.7459766931868
2	-10.4923883983472	13	-4.7459601838597
3	-9.79865984791461	14	-3.7422424168275
4	-8.95687286274882	15	-3.7422420059152
5	-8.61598499608140	16	-2.7072530996296
6	-7.78513533125818	17	-2.7072530926148
7	-7.70309492286509	18	-1.5355155681389
8	-6.75014601406028	19	-1.5355155680825
9	-6.74184019200305	20	-3.25405817949E-8
10	-5.74641173770100	21	-3.25405101953E-8
11	-5.74594979044199		

Relative gaps of the eigenvalues

1/2	0.116	10/11	8.039**E-5**
2/3	0.066	11/12	0.174
3/4	0.086	12/13	3.479**E-6**
4/5	0.038	13/14	0.211
5/6	0.096	14/15	1.100**E-7**
6/7	0.011	15/16	0.277
7/8	0.124	16/17	2.591**E-9**
8/9	1.230E-3	17/18	0.433
9/10	0.148	18/19	3.673**E-11**
		19/20	0.100
		20/21	2.200**E-6**

Representation tree (1)



FOR EACH eigenvalue with a large relative gap

- Compute eigenvalue to high rel. accuracy.
- Compute the FP vector (eigenvector).

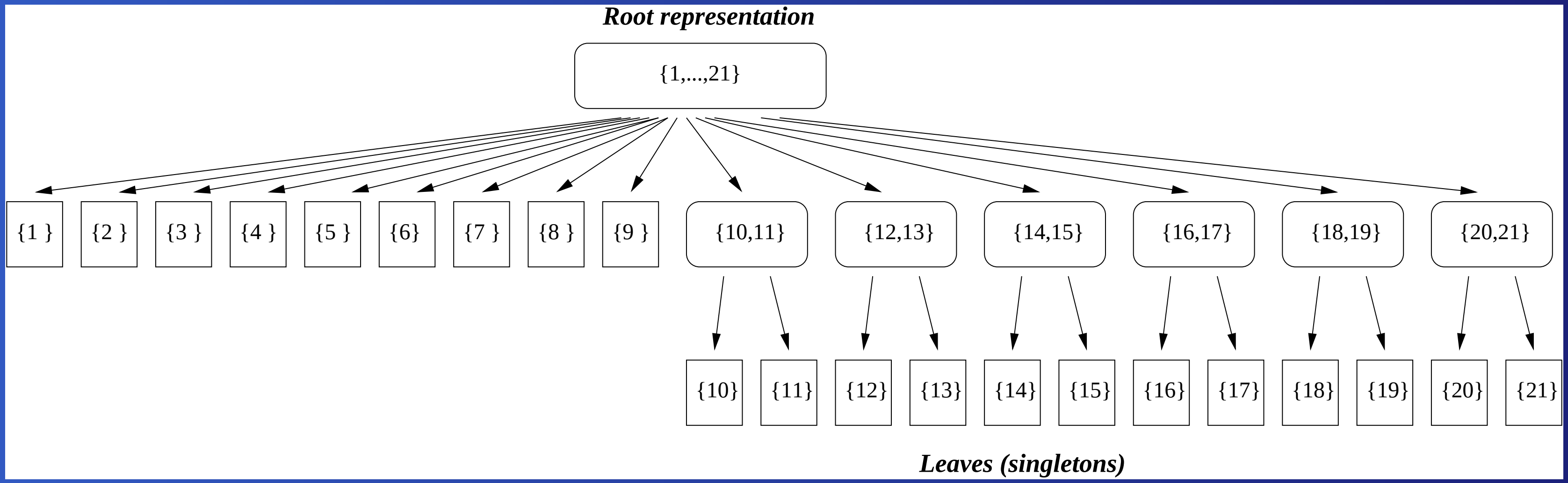
FOR EACH of the remaining groups of eigenvalues

- Choose shift σ outside the group.
- Compute new RRR $L_+D_+L_+^T = LDL^T - \sigma I$.
- Refine the eigenvalues.

Eigenvalues & relgaps, next tree level

10	-4.619472590E-4	1.0	16	-7.014749992E-9	1.0
11	-1.821136709E-15		17	-6.432284273E-17	
12	-1.650932707E-5	1.0	18	-5.641544664E-11	1.0
13	-5.239123482E-15		19	-1.932676195E-15	
14	-4.109123007E-7	1.0	20	-7.159957809E-14	1.0
15	-3.071368719E-17		21	-9.193866613E-23	

Representation tree (final)



How small differences matter

Local $\lambda_{20} = -7.159957809E - 14$.
Local $\lambda_{21} = -9.193866613E - 23$.

- relative gap is 1.
- v_{20} and v_{21} numerically orthogonal
- compare the pivots of $L_+D_+L_+^T = LDL^T - \lambda_i I$

Comparison of $D_+^{(20)}$ and $D_+^{(21)}$

1	-0.746194182903322	-0.746194182903394
2	-0.406060444378103	-0.406060444378303
3	-0.283506635162642	-0.283506635163928
9	-0.102604692636414	-0.104602155218999
10	**-5.125020570206917E-5**	**-0.186161716016229**
11	**19501.370612006347**	**-5.374520449981733**
12	-9.746245461350355	-9.560131070983632
13	-8.643590569648804	-8.641593106560922
19	-2.462687547740681	-2.462687547739467
20	-1.340133738525219	-1.340133738525091
21	4.944470344487295E-23	1.957382746439043E-23

The final slide

Residual: $\max \|(T - \lambda_i I)v_i\| / \|v_i\| = 2.36 n\epsilon \|T\|$

Orthogonality: $\max_{i \neq j} |v_i^T v_j| = 24.47 n\epsilon$