Math 128a - Homework 7 - Due April 18

1) Problem 8.1.9.

Answer: We will use theorem 2 since it works in this case. Our function $f(t,x)$ is

$$f(t,x) = t^2 + e^x$$

and our initial value point (t_0, x_0) is just $(0,0)$. So our goal is to find positive numbers α and β so that, on the rectangle

$$\{(t,x) \mid |t| \leq \alpha, |x| \leq \beta\}$$

we have a constant M such that $|f(t,x)| \leq M$ and a constant C so that

$$\min(\alpha, \frac{\beta}{M}) = C > 0.351 \ \ .$$

Since f and $\partial f/\partial x$ are both continuous for all (t,x), theorem 2 will tell us that there exists a unique solution on the interval $|t| < C$, which in particular will mean that there exists a unique solution on the interval $|t| \leq 0.351$.

First we compute M in terms of α and β. If $|t| < \alpha$ then $t^2 < \alpha^2$ and if $|x| < \beta$ then $e^x < e^\beta$. Thus, on this rectangle,

$$f(t,x) = t^2 + e^x < \alpha^2 + e^\beta = M \ .$$

So now we want to find positive α and β so that

$$\frac{\beta}{\alpha^2 + e^\beta} > 0.351$$

and then we will remember that we also want $\alpha > 0.351$ to get

$$\min(\alpha, \frac{\beta}{\alpha^2 + e^\beta}) > 0.351 \ .$$

The simplest thing to do is to plug in $\beta = 1$ and hope for the best. When we do this we get:

$$\begin{aligned} \frac{\beta}{\alpha^2 + e^\beta} &= \frac{1}{\alpha^2 + e} \\ \frac{1}{\alpha^2 + e} &> 0.351 \\ 1 &> (0.351)\alpha^2 + (0.351)e \\ \alpha^2 &< \frac{1 - (0.351)e}{0.351} \\ \alpha &< \sqrt{\frac{1}{0.351} - e} \\ &= 0.36155\ldots \end{aligned}$$

Thus if we take $\alpha = 0.36$ and $\beta = 1$ we achieve the desired result.

2) Problem 8.2.6.

Answer: Differentiating using the fundamental theorem of calculus we get

$$x'(t) = \cos(t + x(t)) + e^t$$

This is an ordinary differential equation. To get the initial value, note that

$$x(0) = \int_0^0 \cos(s + x(s))ds + e^0 = 0 + 1 = 1 \ .$$

3) Problem 8.3.6.
Answer: Here we just calculate all the terms in equation (8) on page 541, with $f(t, x) = \lambda x$.

$$\begin{aligned}
F_1 &= h\lambda x \\
F_2 &= (h\lambda + \frac{h^2\lambda^2}{2})x \\
F_3 &= (h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{4})x \\
F_4 &= (h\lambda + h^2\lambda^2 + \frac{h^3\lambda^3}{2} + \frac{h^4\lambda^4}{4})x
\end{aligned}$$

and then plug everything in and simplify to get:

$$\begin{aligned}
x(t+h) &= x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4) \\
&= (1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24})x(t)
\end{aligned}$$

4) Problem 8.3.7.
Answer: The point is that we can solve this ODE explicitly. The solution is

$$x(t) = Ae^{\lambda t}$$

for some constant A. Thus

$$x(t+h) = Ae^{\lambda t + \lambda h} = e^{\lambda h}x(t)$$

The Taylor expansion for $e^{\lambda h}$ is

$$e^{\lambda h} = 1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24} + O(h^5)$$

Thus we get

$$x(t+h) = (1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24} + O(h^5))x(t)\ .$$