Math 128a - Homework 7 - Due April 18

1) Problem 8.1.9.

*Answer:* We will use theorem 2 since it works in this case. Our function \( f(t, x) \) is

\[
f(t, x) = t^2 + e^x
\]

and our initial value point \((t_0, x_0)\) is just \((0, 0)\). So our goal is to find positive numbers \( \alpha \) and \( \beta \) so that, on the rectangle

\[
\{(t, x) \mid |t| \leq \alpha, |x| \leq \beta\}
\]

we have a constant \( M \) such that \(|f(t, x)| \leq M\) and a constant \( C \) so that

\[
\min(\alpha, \frac{\beta}{M}) = C > 0.351
\]

Since \( f \) and \( \partial f / \partial x \) are both continuous for all \((t, x)\), theorem 2 will tell us that there exists a unique solution on the interval \(|t| < C\), which in particular will mean that there exists a unique solution on the interval \(|t| \leq 0.351\).

First we compute \( M \) in terms of \( \alpha \) and \( \beta \). If \(|t| < \alpha\) then \( t^2 < \alpha^2 \) and if \(|x| < \beta\) then \( e^x < e^\beta \).

Thus, on this rectangle,

\[
f(t, x) = t^2 + e^x < \alpha^2 + e^\beta = M
\]

So now we want to find positive \( \alpha \) and \( \beta \) so that

\[
\frac{\beta}{\alpha^2 + e^\beta} > 0.351
\]

and then we will remember that we also want \( \alpha > 0.351 \) to get

\[
\min(\alpha, \frac{\beta}{\alpha^2 + e^\beta}) > 0.351
\]

The simplest thing to do is to plug in \( \beta = 1 \) and hope for the best. When we do this we get:

\[
\frac{1}{\alpha^2 + e^\beta} = \frac{1}{\alpha^2 + e}
\]

\[\frac{1}{\alpha^2 + e} > 0.351\]

\[1 > (0.351)\alpha^2 + (0.351)e\]

\[\alpha^2 < \frac{1 - (0.351)e}{0.351}\]

\[\alpha < \sqrt{\frac{1}{0.351} - e} = 0.36155\ldots\]

Thus if we take \( \alpha = 0.36 \) and \( \beta = 1 \) we achieve the desired result.

2) Problem 8.2.6.

*Answer:* Differentiating using the fundamental theorem of calculus we get

\[
x'(t) = \cos(t + x(t)) + e^t
\]

This is an ordinary differential equation. To get the initial value, note that

\[
x(0) = \int_0^0 \cos(s + x(s))ds + e^0 = 0 + 1 = 1.
\]
3) Problem 8.3.6.
Answer: Here we just calculate all the terms in equation (8) on page 541, with $f(t, x) = \lambda x$.

\[
\begin{align*}
F_1 &= h\lambda x \\
F_2 &= (h\lambda + \frac{h^2\lambda^2}{2})x \\
F_3 &= (h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{4})x \\
F_4 &= (h\lambda + h^2\lambda^2 + \frac{h^3\lambda^3}{2} + \frac{h^4\lambda^4}{4})x
\end{align*}
\]

and then plug everything in and simplify to get:

\[
x(t + h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4) \\
= (1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24})x(t)
\]

4) Problem 8.3.7.
Answer: The point is that we can solve this ODE explicitly. The solution is

\[
x(t) = Ae^{\lambda t}
\]

for some constant $A$. Thus

\[
x(t + h) = Ae^{\lambda(t+h)} = e^{\lambda h}x(t)
\]

The Taylor expansion for $e^{\lambda h}$ is

\[
e^{\lambda h} = 1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24} + O(h^5)
\]

Thus we get

\[
x(t + h) = (1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24} + O(h^5))x(t)
\]