Math 128a - Homework 6 - Due April 11

1) Let $[a, b]$ be any closed, bounded interval. Let $x_1, ..., x_n$ be the roots of the Chebyshev polynomial $T_n(x)$, and let $z_i = (x_i + 1) * (b - a)/2 + a$ for $i = 1, ..., n$. Show that of all possible choices of n interpolation points in $[a, b]$, the choice $z_1, ..., z_n$ minimizes the maximum value of the following factor of the interpolation error:

$$\|\prod_{i=1}^{n}(x - z_i)\|_\infty \equiv \max_{a \le x \le b} |\prod_{i=1}^{n}(x - z_i)|$$

and in fact makes

$$\|\prod_{i=1}^{n}(x - z_i)\|_\infty = (b - a)^n 2^{1-2n} \quad .$$

Answer: Let

$$\begin{aligned} p(x) &= \prod_{i=1}^{n}(x - x_i) \quad \text{on } [-1, 1], \text{and let :} \\ q(z) &= \prod_{i=1}^{n}(z - z_i) \quad \text{on } [a, b] \end{aligned}$$

Note that, if $z = \frac{x+1}{2}(b - a) + a$ then $z - z_i = \frac{b-a}{2}(x - x_i)$. Thus $q(z) = (\frac{b-a}{2})^n p(x)$. Also, $x \in [-1, 1]$ if and only if $z \in [a, b]$ (z is a linear function of x and when $x = -1$, we get $z = a$, when $x = 1$ we get $z = b$). We know that

$$\|p\|_\infty = 2^{1-n}$$

so thus we can conclude that

$$\|q\|_\infty = \left(\frac{b-a}{2}\right)^n 2^{1-n} = (b - a)^n 2^{1-2n}$$

Now we just need to show that, if $q_1(z)$ is any other *monic* degree n polynomial on $[a, b]$, then

$$\|q_1\|_\infty \ge \left(\frac{b-a}{2}\right)^n 2^{1-n}$$

To see this, let $p_1(x) = (\frac{2}{b-a})^n q_1(\frac{x+1}{2}(b - a) + a)$ on $[-1, 1]$. The polynomial p_1 is monic because q_1 is monic and degree n, so that the coefficient of x^n in $q_1(\frac{x+1}{2}(b - a) + a)$ is $(\frac{b-a}{2})^n$. Thus:

$$\|p_1\|_\infty \ge 2^{1-n}$$

and

$$\|q_1\|_\infty \ge \left(\frac{b-a}{2}\right)^n 2^{1-n} = \|q\|_\infty$$

2) Problem 7.1.6.

Answer: This is brute force calculation, expanding all the Taylor series out to $O(h^6)$. (We only need $O(h^5)$ for the first one, but we will need $O(h^6)$ for the second one so we might as well do it.) First we expand in h:

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + f''(x)\frac{h^2}{2} + f^{(3)}(x)\frac{h^3}{3!} + f^{(4)}(x)\frac{h^4}{4!} + f^{(5)}(x)\frac{h^5}{5!} + O(h^6) \\ f(x-h) &= f(x) - f'(x)h + f''(x)\frac{h^2}{2} - f^{(3)}(x)\frac{h^3}{3!} + f^{(4)}(x)\frac{h^4}{4!} - f^{(5)}(x)\frac{h^5}{5!} + O(h^6) \end{aligned}$$

and add and subtract (and simplify) to get:

$$\begin{aligned} f(x+h) - f(x-h) &= 2[f'(x)h + f^{(3)}(x)\frac{h^3}{6} + O(h^5)] \\ f(x+h) + f(x-h) &= 2[f(x) + f''(x)\frac{h^2}{2} + f^{(4)}(x)\frac{h^4}{24} + O(h^6)] \end{aligned}$$

Then we expand in 2h:

$$\begin{aligned} f(x+2h) &= f(x) + f'(x)(2h) + f''(x)\frac{(2h)^2}{2} + f^{(3)}(x)\frac{(2h)^3}{3!} + f^{(4)}(x)\frac{(2h)^4}{4!} + f^{(5)}(x)\frac{(2h)^5}{5!} + O(h^6) \\ f(x-2h) &= f(x) - f'(x)(2h) + f''(x)\frac{(2h)^2}{2} - f^{(3)}(x)\frac{(2h)^3}{3!} + f^{(4)}(x)\frac{(2h)^4}{4!} - f^{(5)}(x)\frac{(2h)^5}{5!} + O(h^6) \end{aligned}$$

and add and subtract (and simplify) to get:

$$\begin{aligned} f(x+2h) - f(x-2h) &= 2[f'(x)(2h) + f^{(3)}(x)\frac{4h^3}{3} + O(h^5)] \\ f(x+2h) + f(x+2h) &= 2[f(x) + f''(x)(2h^2) + f^{(4)}(x)\frac{2h^4}{3} + O(h^6)] \end{aligned}$$

Now for the approximation of $f'(x)$ we make the following calculation:

$$8[f(x+h) - f(x-h)] - [f(x+2h) - f(x-2h)] = 12f'(x)h + O(h^5)$$

so that:

$$f'(x) = \frac{1}{12h}\left[8[f(x+h) - f(x-h)] - [f(x+2h) - f(x-2h)]\right] + O(h^4)$$

Likewise for $f''(x)$ we calculate as follows:

$$16[f(x+h) + f(x-h)] - [f(x+2h) + f(x-2h)] = 30f(x) + 16f''(x)h^2 + O(h^6)$$

so that

$$f''(x) = \frac{1}{12h^2}\left[16[f(x+h) + f(x-h)] - [f(x+2h) + f(x-2h)] - 30f(x)\right] + O(h^4)$$

and we're done.

3) Problem 7.2.1.

Answer: The Newton-Cotes formula in this case will be:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{3} A_i f(x_i)$$

where $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$ and $x_3 = 1$. We just have to compute A_0, A_1, A_2, A_3, which we do using the method of undetermined coefficients, using the trial functions $f(x) = 1, x, x^2, x^3$. We get:

$$\begin{aligned}
1 = \int_0^1 dx &= A_0 + A_1 + A_2 + A_3 \\
\frac{1}{2} = \int_0^1 x dx &= \frac{1}{3}A_1 + \frac{2}{3}A_2 + A_3 \\
\frac{1}{3} = \int_0^1 x^2 dx &= \frac{1}{9}A_1 + \frac{4}{9}A_2 + A_3 \\
\frac{1}{4} = \int_0^1 x^3 dx &= \frac{1}{27}A_1 + \frac{8}{27}A_2 + A_3
\end{aligned}$$

or, in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{9} & \frac{4}{9} & 1 \\ 0 & \frac{1}{27} & \frac{8}{27} & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

You can solve this quickly in matlab to get

$$\begin{aligned}
A_0 = 0.1250 &= \frac{1}{8} \\
A_1 = 0.3750 &= \frac{3}{8} \\
A_2 = 0.3750 &= \frac{3}{8} \\
A_3 = 0.1250 &= \frac{1}{8}
\end{aligned}$$

We could have done it by hand with Gaussian elimination, and in fact we should really check these “exact” answers because they were obtained numerically - plugging them in to the matrix equation we find that they do in fact work exactly.

4) Problem 7.5.2.

Answer: Starting with

$$\int_u^v f(x)dx = T(u,v) - \frac{1}{12}(v-u)^3 f''(\xi)$$

where

$$T(u,v) = \frac{1}{2}(v-u)[f(u)+f(v)]$$

we let w be the midpoint $w = (u+v)/2$ and proceed as in the text:

$$\begin{aligned}
\int_u^v f(x)dx &= \int_u^w f(x)dx + \int_w^v f(x)dx \\
&= T(u,w) - \frac{1}{12}(w-u)^3 f''(\xi_1) + T(w,v) - \frac{1}{12}(v-w)^3 f''(\xi_2) \\
&= T^* + T^{**} - \frac{1}{12}\left(\frac{v-u}{2}\right)^3 [f''(\xi_1) + f''(\xi_2)] \\
&= T^* + T^{**} - \frac{1}{48}(v-u)^3 f''(\xi_3)
\end{aligned}$$

where $x_1 \in (u,w)$, $\xi_2 \in (w,v)$, $\xi_3 \in (u,v)$ and

$$\begin{aligned}
T^* &= T(u,w) \\
T^{**} &= T(w,v) \\
f''(\xi_3) &= \frac{1}{2}[f''(\xi_1) + f''(\xi_2)]
\end{aligned}$$

(using the intermediate value theorem as usual for f'').

Now we have to assume that f'' is relatively flat on our interval $[u,v]$ so that $f''(\xi) = f''(\xi_3)$ and we combine the following two equations:

$$\begin{aligned}
\int_u^v f(x)dx &= T(u,v) - \frac{1}{12}(v-u)^3 f''(\xi) \\
\int_u^v f(x)dx &\approx T^* + T^{**} - \frac{1}{48}(v-u)^3 f''(\xi)
\end{aligned}$$

to get:

$$\begin{aligned}
3\int_u^v f(x)dx &\approx 4[T^* + T^{**} - \frac{1}{48}(v-u)^3 f''(\xi)] - \\
&\quad [T(u,v) - \frac{1}{12}(v-u)^3 f''(\xi)] \\
&= 4(T^* + T^{**}) - T(u,v)
\end{aligned}$$

and thus:

$$\begin{aligned}
\int_u^v f(x)dx &\approx \frac{4}{3}(T^* + T^{**}) - \frac{1}{3}T(u,v) \\
&= T^* + T^{**} + \frac{1}{3}[T^* + T^{**} - T(u,v)]
\end{aligned}$$

5) Show that

$$\int_1^{\infty} \frac{\cos(x^2)}{\sqrt{|\sin(x^2)| \cdot x^{1.1}}} dx$$

is finite and compute it as accurately as you can, using Matlab, including the quad command. Explain how you did the integral, and how accurate you think your answer is. Hint: This integral has numerous places ("singularities") where the integrand goes to infinity. Break the interval up into small intervals where the singularities occur at the endpoints, and handle each interval separately. Try changing variables to remove the singulatities. Argue that you can bound the error by integrating from 1 to some large finite value instead of to infinity (subhint: alternating series).

Answer:

Observe that, on the domain of integration, the integrand is singular when x^2 is an integer multiple of π. As a first step, then, we write the integral as

$$\begin{aligned}
\int_1^{\infty} \frac{\cos(x^2)}{\sqrt{|\sin(x^2)| \cdot x^{1.1}}}\, dx = & \int_1^{\sqrt{\pi}} \frac{\cos(x^2)}{\sqrt{|\sin(x^2)| \cdot x^{1.1}}}\, dx \\
& + \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \frac{\cos(x^2)}{\sqrt{|\sin(x^2)| \cdot x^{1.1}}}\, dx \\
& + \int_{\sqrt{2\pi}}^{\sqrt{3\pi}} \frac{\cos(x^2)}{\sqrt{|\sin(x^2)| \cdot x^{1.1}}}\, dx + \dots
\end{aligned}$$

Now we note that on each of the intervals in the above decomposition, $\sin(x^2)$ is either always nonnegative or always nonpositive. Hence, we can dispense with the absolute value signs by writing

$$\begin{aligned}
&\int_1^{\infty} \frac{\cos(x^2)}{\sqrt{|\sin(x^2)| \cdot x^{1.1}}}\, dx \\
&\qquad = \int_1^{\sqrt{\pi}} \frac{\cos(x^2)}{\sqrt{\sin(x^2) \cdot x^{1.1}}}\, dx + \sum_{n=1}^{\infty} \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \frac{\cos(x^2)}{\sqrt{(-1)^n \sin(x^2) \cdot x^{1.1}}}\, dx.
\end{aligned}$$

We will now rid ourselves of the singularities through the judicious use of a substitution. We wish to make the substitution $u = \sqrt{(-1)^n \sin(x^2)}$, where the n corresponds to the intervals in the previous sum. However, we are only allowed to make this substitution on intervals on which $\sin(x^2)$ is one-to-one. To satisfy this requirement, we must divide the intervals again, writing

$$\int_1^{\infty} \frac{\cos(x^2)}{\sqrt{|\sin(x^2)| \cdot x^{1.1}}}\, dx = I_1 + I_2 + I_3 + \sum_{k=1}^{\infty} \left[H_1(k) + H_2(k) + H_3(k) + H_4(k) \right],$$

where

$$I_1 = \int_1^{\sqrt{\pi/2}} \frac{\cos(x^2)}{\sqrt{\sin(x^2) \cdot x^{1.1}}}\, dx, \quad I_2 = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \frac{\cos(x^2)}{\sqrt{\sin(x^2) \cdot x^{1.1}}}\, dx,$$

$$I_3 = \int_{\sqrt{\pi}}^{\sqrt{3\pi/2}} \frac{\cos(x^2)}{\sqrt{-\sin(x^2) \cdot x^{1.1}}}\, dx,$$

$$H_1(k) = \int_{\sqrt{(4k-1)\pi/2}}^{\sqrt{2k\pi}} \frac{\cos(x^2)}{\sqrt{-\sin(x^2) \cdot x^{1.1}}}\, dx, \quad H_2(k) = \int_{\sqrt{2k\pi}}^{\sqrt{(4k+1)\pi/2}} \frac{\cos(x^2)}{\sqrt{\sin(x^2) \cdot x^{1.1}}}\, dx,$$

$$H_3(k) = \int_{\sqrt{(4k+1)\pi/2}}^{\sqrt{(2k+1)\pi}} \frac{\cos(x^2)}{\sqrt{\sin(x^2) \cdot x^{1.1}}}\, dx, \quad H_4(k) = \int_{\sqrt{(2k+1)\pi}}^{\sqrt{(4k+3)\pi/2}} \frac{\cos(x^2)}{\sqrt{-\sin(x^2) \cdot x^{1.1}}}\, dx.$$

(Note that $I_2 = H_3(0)$ and $I_3 = H_4(0)$.)

Now we make the substitution

$$u = \sqrt{\sin(x^2)}, \quad du = \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}\,dx$$

in the integrals $I_1, I_2, H_2(k)$, and $H_3(k)$; and the substitution

$$u = \sqrt{-\sin(x^2)}, \quad du = \frac{-x\cos(x^2)}{\sqrt{\sin(x^2)}}\,dx$$

in the integrals $I_3, H_1(k)$, and $H_4(k)$. When we make these substitutions, we need to solve for x using the arcsin function. Because the range of arcsin is $[-\pi/2, \pi/2]$, we must "shift" the values to properly invert the substitution. We get

$$x = \begin{cases} \sqrt{\arcsin(u^2)} & \text{in } I_1, \\ \sqrt{\pi - \arcsin(u^2)} & \text{in } I_2, \\ \sqrt{\pi - \arcsin(-u^2)} & \text{in } I_3, \\ \sqrt{2k\pi + \arcsin(-u^2)} & \text{in } H_1(k), \\ \sqrt{2k\pi + \arcsin(u^2)} & \text{in } H_2(k), \\ \sqrt{(2k+1)\pi - \arcsin(u^2)} & \text{in } H_3(k), \\ \sqrt{(2k+1)\pi - \arcsin(-u^2)} & \text{in } H_4(k). \end{cases}$$

Making these substitutions, we get

$$\begin{aligned} I_1 &= \int_{\sqrt{\sin(1)}}^{1} \frac{1}{(\arcsin(u^2))^{1.05}}\,dx, \\ I_2 &= -\int_0^1 \frac{1}{(\pi - \arcsin(u^2))^{1.05}}\,du, \\ I_3 &= -\int_0^1 \frac{1}{(\pi - \arcsin(-u^2))^{1.05}}\,du, \\ H_1(k) &= \int_0^1 \frac{1}{(2k\pi + \arcsin(-u^2))^{1.05}}\,du, \\ H_2(k) &= \int_0^1 \frac{1}{(2k\pi + \arcsin(u^2))^{1.05}}\,du, \\ H_3(k) &= -\int_0^1 \frac{1}{((2k+1)\pi - \arcsin(u^2))^{1.05}}\,du, \\ H_4(k) &= -\int_0^1 \frac{1}{((2k+1)\pi - \arcsin(-u^2))^{1.05}}\,du. \end{aligned}$$

None of these integrals have singularities.

The only remaining difficulty is in showing that the infinite series $\sum_{k=1}^{\infty}[H_1(k) + H_2(k) + H_3(k) + H_4(k)]$ converges. Since $H_1(k)$ and $H_2(k)$ are positive, and $H_3(k)$ and $H_4(k)$ are negative, this is an alternating series. Therefore, it converges if and only if the sequence $\{|H_1(k) + H_2(k)|, |H_3(k) + H_4(k)|\}_{k=1}^{\infty}$ decreases to zero. To show that this is true, observe that the integrands of $H_1(k)$ and $H_2(k)$ are both bounded above by $((2k - \frac{1}{2})\pi)^{-1.05}$. Therefore,

$$|H_1(k) + H_2(k)| \le 2((2k - \tfrac{1}{2})\pi)^{-1.05}.$$

Similarly,

$$|H_3(k) + H_4(k)| \le 2((2k + \tfrac{1}{2})\pi)^{-1.05}.$$

From these bounds it is easy to see that the sequence $\{|H_1(k) + H_2(k)|\,, |H_3(k) + H_4(k)|\}_{k=1}^{\infty}$ decreases to zero as desired, so the series converges. This proves that $\int_1^\infty \cos(x^2)(\sqrt{|\sin(x^2)|} \cdot x^{1.1})^{-1}\,dx$ exists.

Now we must estimate the value of this integral. Let I denote its true value. Estimating I involves estimating the values of $I_1, I_2, I_3, H_1(k), H_2(k), H_3(k)$, and $H_4(k)$, and estimating the value of the series $\sum_{k=1}^{\infty}[H_1(k) + H_2(k) + H_3(k) + H_4(k)]$. The former can be accomplished using the `quad` command from MATLAB; to estimate the series, we will use a partial sum, as described in more detail below. We see therefore that we can estimate I to within an absolute error of ϵ is we make sure that the total quadrature error is less than $\epsilon/2$ and that the error in estimating the series is less than $\epsilon/2$.

Since the series is alternating, we know that the error in estimating its value with a partial sum is bounded by the absolute value of the first neglected term. Hence, to estimate the series within an error of $\epsilon/2$, it is enough to find an N such that

$$|H_1(N) + H_2(N)| < \frac{\epsilon}{2};$$

Finding such an N can be done using the bounds on $|H_1(k) + H_2(k)|$ that we obtained above. Once we have found N, we estimate the sum of the series with the partial sum

$$\sum_{k=1}^{N}[H_1(k) + H_2(k) + H_3(k) + H_4(k)].$$

Unfortunately, we can only estimate the value of this partial sum, as each $H_i(k)$ must be computed with MATLAB's `quad` command. When we use quadrature to estimate this sum and the integrals I_1, I_2, and I_3, we will make $4N + 3$ calls to `quad`. Therefore, the total quadrature error is bounded by $\epsilon/2$ if we ensure that the error in each call to `quad` is bounded by $\epsilon/(2(4N + 3))$.

A MATLAB program that implements these ideas to estimate I is given below. The output of the program for given error tolerances is given in the following table. Also included in the table is the value of N necessary to achieve the desired error tolerance. Note that these N values quickly get very large. This has the unfortunate consequence that computing the integral to a high degree of accuracy is impractical (at least with the program below); `integral(1.e-8)` would probably require several days to run.

tol	N	`integral(`*tol*`)`
1.e-1	5	-0.39539048605573
1.e-2	49	-0.37225827183131
1.e-3	443	-0.37003842273182
1.e-4	3972	-0.36981649283492
1.e-5	35600	-0.36979479803630
1.e-6	319029	-0.36979262477282

The reader will see that in the program below we pass the functions `fun1`, `fun2`, `fun3`, and `fun4` to `quad`; these four programs simply return values of the integrands in the corresponding $H_i(k), 1 \le i \le 4$.

```
function i = integral(tol)
% integral(tol) uses quadrature to estimate the integral given in
% Problem 5 of Homework 6, Math 128A, Prof. Demmel, Spring 2002
% We estimate the integral to within an absolute error of tol.

% N determines how many terms of the partial sum we need to compute.
% N is big enough that the first neglected term of the alternating series
% is smaller than tol/2.  The following algorithm for finding N is
% adapted from an earlier homework; since N is roughly on the order of 1/tol,
```

```
% the "increment by 1 and test" algorithm for finding N will be too slow
% for small values of tol.
N = 1;
inc = floor(1/tol);
while(inc > 0)
   if ((2/((2*(N+inc)-0.5)*pi))^(1.05) < tol/2)
      inc = floor(inc/2);
   elseif (N + inc == N)
      inc = 0;
   else
      N = N + inc;
   end
end

% For each value of k from 1 to N in the for-loop below, we calculate four
% integrals using the MATLAB quad command.  We also compute 3 inital
% integrals before we enter the for-loop.  We therefore compute 4N+3
% integrals via quadrature, so the tolerance for each call to quad should
% be (tol/2)/(4N+3).
quadtol = (tol/2)/(4*N+3);

% Add the first three terms before beginning the for-loop for the general
% pattern.
i = 0;
i = i + quad(@fun2,sqrt(sin(1)),1,quadtol,[],0);
i = i + quad(@fun3,0,1,quadtol,[],0);
i = i + quad(@fun4,0,1,quadtol,[],0);

% Now add the remaining terms up to the first neglectable term.
for k=1:N
   i = i + quad(@fun1,0,1,quadtol,[],k);
   i = i + quad(@fun2,0,1,quadtol,[],k);
   i = i + quad(@fun3,0,1,quadtol,[],k);
   i = i + quad(@fun4,0,1,quadtol,[],k);
end
```